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Linear

$\mathbb{R}$ = Set of all real number

$\pi \in \mathbb{R}$ ✓ , $\frac{2}{3} \in \mathbb{R}$ ✓ , as a picture $\longleftarrow$ -2 -1 0 1 2 $\longrightarrow$

$\in$ ↓ belongs

$(2,3) \neq (3,2)$ can't be switched

XY plane we look at points not #s, and each points consists of 2 co-ord

$\mathbb{R}^2$ = Set of all points in the plane

y (2,3) (x,y) x xy plane

$5 \in \mathbb{R}^2$ X , $(-5,2) \in \mathbb{R}^2$ ✓ , $(0,3) \in \mathbb{R}$ X

↓ # not a point → such that

$\mathbb{R}^2 = \underbrace{\{(x_1, x_2) \mid x_1, x_2 \in \mathbb{R}\}}_{\text{set}}$

$\mathbb{R}^3 = \{(x_1, x_2, x_3) \mid x_1, x_2, x_3 \in \mathbb{R}\}$
xyz space

$\mathbb{R}^4 = \{(x_1, x_2, x_3, x_4) \mid x_1, x_2, x_3, x_4 \in \mathbb{R}\}$

let n be a +ve $\underbrace{\text{integer}}_{\text{whole \#}}$

$\mathbb{R}^n = \{(x_1, x_2, \dots x_n) \mid x_1, x_2, \dots x_n \in \mathbb{R}\}$

Solve the system

$\left.\begin{aligned} x_1 + x_2 &= 8 \\ 2x_1 + 2x_2 &= 4 \end{aligned}\right\}$ N (m,n) M → we can have two lines and they intersect each line lives in the plane $\mathbb{R}^2$

$x_2 = 6$ $x_1 = 2$ (2,6) → one soln

Find the solution to the system

$\left.\begin{aligned} x_1 + x_2 &= 8 \\ -x_1 + x_2 &= 4 \end{aligned}\right\}$ # we have two unknowns (variables) ∴ solns set must live in $\mathbb{R}^2$ {variables not # of eqn determines the soln set}

$x_2 = 6$ $x_1 = 2$ → not a soln set

$\{(2,6)\}$ lives in $\mathbb{R}^2$ → The soln is unique → one soln

method for any # of eqns and # of unknowns

5var, 6eqn → Soln in $\mathbb{R}^5$

← unique
{no more than one soln for linear (power 1) 2D}

no soln: $\{\ \} = \emptyset$

set ↓ unique

$x_1 + x_2 = 1$
$2x_1 + 2x_2 = 2$
∞ many solns

If I found 2 points that satisfy the eqn → then ∞ many solns

we can find ∞ → #s with restrictions on variables

3x4 system of linear eqns (The order is important)
↳ #variables
↳ # of eqns

Ex.

$$\begin{aligned} X_1 - 2X_2 + X_4 &= 0 \quad (\rightarrow 0X_3) \\ -X_1 + 2X_2 - X_3 &= 10 \\ X_1 + 2X_2 + X_3 + X_4 &= 12 \end{aligned} \Bigg\} \text{3 eqns}$$

↳ 4 variables

3 x 4 system of linear eqn
↳ all variables to the power 1

Soln set it lives in $\mathbb{R}^4$ {Same as the # of variables}
↳ point consists of 4 co-ordinates

I can multiply one of the eqns by a constant → The look will be different but the soln is the same (constant should be non-zero)
Swapping → " " " "
Multiply and add the sides → " " " → The co-ef changes, nothing else changes

$X_1 - 2X_2 = 4$) X -3
$3X_1 - X_2 = 2$

$-3X_1 + 6X_2 + 3X_1 - X_2 = -12 + 2$

Ex. Find the soln set of the following system

$$\begin{aligned} X_1 - X_3 &= 0 \\ -X_1 + X_2 + X_3 &= 1 \\ -2X_1 - 4X_2 + 2X_3 &= -4 \end{aligned}$$

$\mathbb{R}^3$ ⇝ if $(0,3,1)$ was the answer, then if I substituted the (X_1, X_2, X_3) the three eqns should be satisfied
↳ it doesn't satisfy the eqns so not a soln

AUGMENTED METHOD → The best method to solve, cause the other methods have restrictions. Here the only restriction is to be a linear eqn

co-eff

$$\begin{array}{cccc} X_1 & X_2 & X_3 & C \\ \end{array}$$
$$\left[\begin{array}{ccc|c} 1 & 0 & -1 & 0 \\ -1 & 1 & 1 & 1 \\ -2 & -4 & 2 & -4 \end{array}\right] \leftarrow \text{augmented}$$

Isolate as many variables as possible {ISOLATE AS MAXIMUM WE CAN}

Method : Work row by row

I) First row :

a) make the 1st nonzero # be "1" → Leader

b) use the leader to kill all #s exactly below and above the leader

in this question no above, it's the 1st row 1

Row operation

1) αR_i , $\alpha \neq 0$

multiply i^{th} row with a real #, $\alpha \neq 0$

→ We use it to get the leader.

2) $\alpha R_i + R_k \to R_k$

$3R_i + R_k \to R_k$ → actual change

$-2R_2 + R_1 \to R_1$ → Multiply R_2 with -2 add it to R_1 and the actual change will be in R_1

→ We use it to "Kill" #s exactly above (below) a leader

3) $R_i \xleftrightarrow{\text{interchange}} R_k$

$$\begin{array}{c} \begin{array}{ccc|c} x_1 & x_2 & x_3 & c \end{array} \\ \left[\begin{array}{ccc|c} 0 & 2 & -1 & 0 \\ -1 & 1 & 1 & 1 \\ -2 & -4 & 2 & -4 \end{array}\right] \end{array}$$

This will be one.

$$\begin{array}{c} \begin{array}{ccc|c} x_1 & x_2 & x_3 & c \end{array} \\ \left[\begin{array}{ccc|c} +1 & 0 & -1 & 0 \\ -1 & 1 & 1 & 1 \\ -2 & -4 & 2 & -4 \end{array}\right] \end{array}$$

$1R_1 + R_2 \to R_2$

$2R_1 + R_3 \to R_3$

$$\begin{array}{c|c} & \begin{array}{ccc|c} x_1 & x_2 & x_3 & c \end{array} \\ \begin{array}{c} \\ R_2 \\ R_3 \end{array} & \left[\begin{array}{ccc|c} 1 & 0 & -1 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & -4 & 0 & -4 \end{array}\right] \end{array}$$

new rows → R_2

old rows to be multiplied and added to get the new row

II) Second row :

repeat (a) and (b)

→ { already zero, then make above and below zero's }

$4R_2 + R_3 \to R_3$ ⇝ equivilant, not equal but the same soln set.

$$\left[\begin{array}{ccc|c} 1 & 0 & -1 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

III) 3rd row
repeat (a) and (b)

If it was $\left[\begin{array}{ccc|c} 1 & 0 & -1 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 4 \end{array}\right]$ → Stop, we isolate the variables not the constant (4)

$\left[\begin{array}{ccc|c} 1 & 0 & -1 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{array}\right]$ ⇒ Stop ⇐ Read → $x_1 - x_3 = 0$
$x_2 = 1$

* Leading Variables they correspond to the leaders

* all other variables we call them free. x_3 is free ∴
$x_3 \in \mathbb{R}$ → Can be any real #
→ ∞ many solns

* We write out solns in terms of the free variables

Solve for x_1: $x_1 = x_3$
Solve for x_2: $x_2 = 1$ → If we stop here we get $\boxed{\frac{8}{10}}$
Solution set $\{(x_3, 1, x_3) \mid x_3 \in \mathbb{R}\}$ → lies in $\mathbb{R}^3$

Is the following a soln? (1,1,1)? yes, lives in the soln set ✓
(-3,1,-3)? ✓
(5,1,0)? X

We say a system of linear eqns is consistent if it has a soln (Can be 1 or ∞ many) {the Q: Is the system consistent?}
We say a system of linear eqns is inconsistent if it has no soln

1. If the system is consistent and no free variables then the system will have a unique soln
2. If the system is consistent and we have ~~different~~ free variables then the system has ∞ many solns

$\left[\begin{array}{ccc|c} 1 & 0 & 0 & 2 \\ 0 & 1 & 0 & 4 \\ 0 & 0 & 1 & 5 \\ 0 & 0 & 0 & 10 \end{array}\right]$ → $0 \neq 10$ → inconsistent

Ex. Find a soln set:

$$x_1 - x_2 + x_3 = 1$$
$$-x_1 + x_2 - x_3 = 4$$

Soln:

$$\begin{array}{c} \begin{array}{cccc} x_1 & x_2 & x_3 & c \end{array} \\ \left[\begin{array}{ccc|c} 1 & -1 & 1 & 1 \\ -1 & 1 & -1 & 4 \end{array}\right] \end{array} \leftarrow \text{augmented}$$

$R_1 + R_2 \to R_2$

$$\left[\begin{array}{ccc|c} 1 & -1 & 1 & 1 \\ 0 & 0 & 0 & 5 \end{array}\right] \Rightarrow \text{Stop} \rightarrow \begin{array}{l} x_1 - x_2 + x_3 = 1 \\ 0 \neq 5 \end{array}$$

$\therefore$ Solution set $= \{ \ \} = \emptyset$ empty set
no soln

Ex. Find the solution set.

$$x_3 - x_4 = 0$$
$$x_1 - x_4 = 1$$
$$x_2 - x_4 = 2$$
$$x_1 + x_2 - x_3 = 4$$

4x4 system

$$\begin{array}{c} \begin{array}{ccccc} x_1 & x_2 & x_3 & x_4 & c \end{array} \\ \left[\begin{array}{cccc|c} 0 & 0 & 1 & -1 & 0 \\ 1 & 0 & 0 & -1 & 1 \\ 0 & 1 & 0 & -1 & 2 \\ 1 & 1 & -1 & 0 & 4 \end{array}\right] \end{array}$$

if x_3 was 2 then I have to multiply the eqn with 1/2 to make the leading variable 1

$R_1 + R_4 \to R_4$

$$\left[\begin{array}{cccc|c} 0 & 0 & 1 & -1 & 0 \\ 1 & 0 & 0 & -1 & 1 \\ 0 & 1 & 0 & -1 & 2 \\ 1 & 1 & 0 & -1 & 4 \end{array}\right]$$

$-R_2 + R_4 \to R_4$

$$\left[\begin{array}{cccc|c} 0 & 0 & 1 & -1 & 0 \\ 1 & 0 & 0 & -1 & 1 \\ 0 & 1 & 0 & -1 & 2 \\ 0 & 1 & 0 & 0 & 3 \end{array}\right]$$

$-R_3 + R_4 \to R_4$

$$\left[\begin{array}{cccc|c} 0 & 0 & 1 & -1 & 0 \\ 1 & 0 & 0 & -1 & 1 \\ 0 & 1 & 0 & -1 & 2 \\ 0 & 0 & 0 & 1 & 1 \end{array}\right]$$

Cont.

$R_4 + R_1 \to R_1$
$R_4 + R_2 \to R_2$
$R_4 + R_3 \to R_3$

$$\left[\begin{array}{cccc|c} 0 & 0 & 1 & 0 & 1 \\ 1 & 0 & 0 & 0 & 2 \\ 0 & 1 & 0 & 0 & 3 \\ 0 & 0 & 0 & 1 & 1 \end{array}\right] \Rightarrow \text{Stop}$$

$$\left.\begin{array}{l} x_3 = 1 \\ x_1 = 2 \\ x_2 = 3 \\ x_4 = 1 \end{array}\right\} \rightarrow \text{unique soln, one point}$$

Soln set $= \{(2,3,1,1)\}$

No solution: inconsistant only if at some point one of the eqns read zero ≠ non zero # (→ Equivilant system (this apply in any step not the final))

If one of the steps reads 0 0 0 | # → we directly stop

Q:

$$\begin{array}{c} \begin{array}{cccc} x_1 & x_2 & x_3 & c \end{array} \\ \left[\begin{array}{ccc|c} 1 & 0 & b & 12 \\ 0 & 1 & 0 & 10 \\ -1 & 0 & 6 & c \end{array}\right] \end{array}$$

* For what values of b, c will the system be consistent?

$R_1 + R_3 \to R_3$ $\left[\begin{array}{ccc|c} 1 & 0 & b & 12 \\ 0 & 1 & 0 & 10 \\ 0 & 0 & 6+b & c+12 \end{array}\right]$ → When I have a variable. I can't change to 1 because if $b=-6$, I can't divide by zero

$x_1 + bx_3 = 12$

$x_2 = 10$

$x_3(6+b) = c+12$

① $b \neq -6$ and $c \in \mathbb{R}$

② $b = -6$ and $c = -12$ → They must be, so the sides be equal, consistant

* For what values of b, c will the system have a unique soln?

Unique → No free variables

$B \neq -6$ and $C \in \mathbb{R}$

Fix B, C we'll have a unique soln → unique depends on B, C values, but for ∞ many solns we should have free variables and be consistent $b=-6$ and $c=-12$

Unique means that we shouldn't " " " " " "

* For what values of b, c will the system have ∞ many solns?

$c = -12$ and $b = -6$ In this case, since we specified b, c we can find the soln set

$$\left.\begin{array}{l} x_1 - 6x_3 = 12 \\ x_2 = 10 \\ 0 = 0 \end{array}\right] \quad \begin{array}{l} x_1 = 12 + 6x_3 \\ x_3 \in \mathbb{R} \end{array}$$

∞ many solns

Soln set $\{(12+6x_3, 10, x_3) \mid x_3 \in \mathbb{R}\}$

(18, 10, 1) ✓ answer

(24, 10, 2) ✓ answer

(22, 10, 2) ✗ not an answer

Q: Imagine ...

$$\begin{array}{c} \begin{array}{cccccc} x_1 & x_2 & x_3 & x_4 & x_5 & c \end{array} \\ \left[\begin{array}{ccccc|c} 0 & 1 & 2 & 0 & 1 & 2 \\ 1 & 0 & -1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 5 & 0 \end{array}\right] \end{array}$$

→ Check the leaders ↘ stop calc.

$\boxed{x_2} + 2x_3 + x_5 = 2 \rightarrow x_2 = 2 - 2x_3 - x_5$

$\boxed{x_1} - x_3 = 1 \rightarrow x_1 = 1 + x_3$

$\boxed{x_4} + 5x_5 = 0 \rightarrow x_4 = -5x_5$

leading $\quad x_3, x_5 \in \mathbb{R}$

Soln set $= \{(1+x_3, 2-2x_3-x_5, x_3, -5x_5, x_5) \mid x_3, x_5 \in \mathbb{R}\}$

generate ∞ many solns by deciding on x_3 and x_5

Q: $x_1 - x_2 + x_3 = 0$
$-x_1 + 2x_2 - x_3 = 0$
$-x_2 + x_3 = 0$

→ All cons. are zero

This kind of systems we call it $\boxed{\text{homogeneous sys}}$

all constants = 0

3x3 system

Claim: Every homogeneous $\boxed{\text{system is consistent}}$ → they at least have 1 soln (0,0,...)

↳ always

Assume unique $\quad$ ∞ many

{(0,0,...0)} soln set

$$\begin{array}{c} \begin{array}{cccc} x_1 & x_2 & x_3 & c \end{array} \\ \left[\begin{array}{ccc|c} 1 & -1 & 1 & 0 \\ -1 & 2 & -1 & 0 \\ 0 & -1 & 1 & 0 \end{array}\right] \end{array} \xrightarrow{R_1+R_2 \rightarrow R_2} \left[\begin{array}{ccc|c} 1 & -1 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & -1 & 1 & 0 \end{array}\right] \xrightarrow[R_2+R_3 \rightarrow R_3]{R_2+R_1 \rightarrow R_2} \left[\begin{array}{ccc|c} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{array}\right]$$

→ Stop, unique soln (homogen.) and no free variables

Soln set = {(0,0,0)}

$D = \text{Span}\{(3,2), (-1,1)\} = \mathbb{R}^2$

↳ Set of all possible $\boxed{\text{linear combination}}$ of (3,2) and (-1,1)

$\alpha_1, \alpha_2 \in \mathbb{R}$ choose any # randomly

$\alpha_1(3,2) + \alpha_2(-1,1)$ ↝ linear combination $\quad$ ↩ live in the span (any combination)

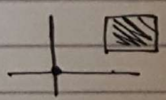
Ex. $\alpha_1 = 0 \; \alpha_2 = 0 \quad 0(3,2) + 0(-1,1) = (0,0) + (0,0) = (0,0)$

$\therefore (0,0) \in D$, (0,0) belong to the given span

∞ combinations

Ex. $\alpha_1 = 1 \; \alpha_2 = -1 \quad 2(3,2) + {}^{-}1(-1,1) = (6,4) + (1,-1) = (7,3) \in D$

Fact (know): Span{point, point, ... point} (0,0,...0) always belong to the given span ↳ In R_4 the span have 4 combinations, 4 αs

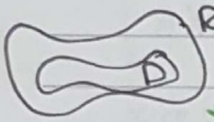
 → Not span, the way drawn doesn't contain zero

As soon D is a set and (0,0,..0) is not D we can't write D as span of some points

we should have (0,0,0,0,0)

$D =$ Soln set $\{(1+x_3, 2-2x_3-x_5, x_3, -5x_5, x_5) \mid x_3, x_5 \in \mathbb{R}\}$

R5 → from # of points



Can we write the soln set of the span?

$x_3 = 0 \quad x_5 = 0 \rightarrow (1,2,0,0,0)$ NO, $(0,0,0,0,0) \notin$ soln set

∴ we can't write it as a span of points.

Q: Imagine

$$\left[\begin{array}{cccc|c} 0 & 1 & 0 & 3 & 0 \\ 1 & 0 & 0 & 5 & 0 \\ 0 & 0 & 1 & 10 & 0 \end{array}\right]$$

→ homogeneous.

→ No more row operations, Stop

$x_2 + 3x_4 = 0 \rightarrow x_2 = -3x_4 \qquad x_4 \in \mathbb{R}$

$x_1 + 5x_4 = 0 \rightarrow x_1 = -5x_4$

$x_3 + 10x_4 = 0 \rightarrow x_3 = -10x_4$

Soln set $= \{(-5x_4, -3x_4, -10x_4, x_4) \mid x_4 \in \mathbb{R}\}$

↓ This set can be written as a span.

$= \{x_4(-5,-3,-10,1) \mid x_4 \in \mathbb{R}\} = \text{Span}\{(-5,-3,-10,1)\}$

↳ Set of all linear combination

If my system is not homogeneous it can't be written as a span at least one eqn .... ≠ Non zero

Q: $D = \text{Span}\{(1,3,0), (-1,2,5)\}$

Linear combination of $5(1,3,0) + -2(-1,2,5) =$

$(5,15,0) + (2,-4,-10) = (7,11,-10) \in D$

We can get ∞ many solns

Fact 1:

If we have a non-homogeneous system, then solution set can not be written as span. Why? Since $(0,0,\dots 0)$ never in the solution set.
depending on # of variables

Fact 2:

Solution set of a homogeneous system can be written as a span {point}. As soon K number of free variables, the soln set = span {Exactly K points}. Soln set never span {# of points < K}.

If the free variables were 7, I can write 7 or more points but never less

Q: Imagine

$D = \text{Soln set} = \{(-2x_4 + 3x_3 - x_5,\ 2x_3 + 2x_5,\ x_3, x_4, x_5) \mid x_3, x_4, x_5 \in \mathbb{R}\}$

of a homogenous system.

Q: Re-write D as a span{...}

I have 3 variables → 3 points

$= \{(x_3(3,2,1,0,0) + x_4(-2,0,0,1,0) + x_5(-1,2,0,0,1) \mid x_3, x_4, x_5 \in \mathbb{R}\}$

$= \text{span}\{(3,2,1,0,0), (-2,0,0,1,0), (-1,2,0,0,1)\}$

Is $(3,2,1,0,0)$ only a solution?

yes, $1 \times (3,2,1,0,0) + 0(-2,0,0,1,0) + 0(-1,2,0,0,1) = (3,2,1,0,0)$

Is $(-1,0,0,\frac{1}{2},0)$ a solution?

yes, $0(3,2,1,0,0) + \frac{1}{2}(-2,0,0,1,0) + 0(-1,2,0,0,1) = (-1,0,0,\frac{1}{2},0)$

Q: Imagine

$\{(0, x_3 + x_4, x_3, x_4, -2x_3 + x_4) \mid x_3, x_4 \in \mathbb{R}\}$ for a homogenous system, write it as a span.

$= \{x_3(0,1,1,0,-2) + x_4(0,1,0,1,1) \mid x_3, x_4 \in \mathbb{R}\}$

$= \text{span}\{(0,1,1,0,-2), (0,1,0,1,1)\}$

$[\quad 1 \,]$ → Augmented: represent a system of linear eqn
→ Without the line, it is a matrix

$A = \begin{bmatrix} 1 & 0 & 4 & 1 \\ -1 & 2 & 0 & 4 \end{bmatrix}$ $B = \begin{bmatrix} 3 & 1 & 2 & 0 \\ -1 & 4 & 5 & 10 \end{bmatrix}$

Size = 2 x 4 Size = 2 x 4

#rows #columns Size is also called order

A+B

$\begin{bmatrix} 4 & 1 & 6 & 1 \\ -2 & 6 & 5 & 14 \end{bmatrix}$

must know A ± B is defined only if size of A = size of B

B - A

$\begin{bmatrix} 2 & 1 & -2 & -1 \\ 0 & 2 & 5 & 6 \end{bmatrix}$

½ A

$\begin{bmatrix} 1/2 & 0 & 2 & 1/2 \\ -1/2 & 1 & 0 & 2 \end{bmatrix}$

4B

$\begin{bmatrix} 12 & 4 & 8 & 0 \\ -4 & 16 & 20 & 40 \end{bmatrix}$

B + 4

undefined, 4 = [4] size 1 x 1

$A + \begin{bmatrix} 1 & 2 \\ 0 & 4 \\ 1 & 1 \end{bmatrix}$

2 x 4 3x2

undefined

How to multiply matrices by linear combination:

$A = \begin{bmatrix} 1 & 2 & -1 \\ 0 & 1 & 4 \end{bmatrix}$ $B = \begin{bmatrix} 1 & 2 \\ -1 & 1 \\ 3 & -4 \end{bmatrix}$ 3X2

2X3

Find AB using linear combination method?

AB = defined = C

2 x 3 3 x 2

3 = 3

size of C 2X2

1st column of $C = AB$ put columns of the 1st matrix. (here A)

$$1\begin{bmatrix}1\\0\end{bmatrix} + -1\begin{bmatrix}2\\1\end{bmatrix} + 3\begin{bmatrix}-1\\4\end{bmatrix} = \begin{bmatrix}-4\\11\end{bmatrix}$$

↳ 1st column of $AB = C$

from B, 1st column we take the scallers

↳ 1, -1, 3 : are 1st column of the 2nd matrix

$$2\begin{bmatrix}1\\0\end{bmatrix} + 1\begin{bmatrix}2\\1\end{bmatrix} + -4\begin{bmatrix}-1\\4\end{bmatrix} = \begin{bmatrix}8\\-15\end{bmatrix}$$

↳ 2nd column of $AB = C$

$$AB = C = \begin{bmatrix}-4 & 8\\11 & -15\end{bmatrix}$$

* No need to solve all the steps, we can solve for one step.

Find BA = defined = $D = 3\times3$

$3\times\boxed{2}$ $\boxed{2}\times 3$

= size

$$1\begin{bmatrix}1\\-1\\3\end{bmatrix} + 0\begin{bmatrix}2\\1\\4\end{bmatrix} = \begin{bmatrix}1\\-1\\3\end{bmatrix}$$

$$2\begin{bmatrix}1\\-1\\3\end{bmatrix} + 1\begin{bmatrix}2\\1\\4\end{bmatrix} = \begin{bmatrix}4\\-1\\2\end{bmatrix}$$

$$-1\begin{bmatrix}1\\-1\\3\end{bmatrix} + 4\begin{bmatrix}2\\1\\4\end{bmatrix} = \begin{bmatrix}7\\5\\-19\end{bmatrix}$$

$$BA = \begin{bmatrix}1 & 4 & 7\\-1 & -1 & 5\\3 & 2 & -19\end{bmatrix}$$

* A, B are equiuilant means one of them is obtained from the other using row operations
* A = B means A, B are identical

Linear combination method

Common sense properties

1. $AB = CD$

$FAB = FCD$ order is imp. but $FAB \neq CDF$

$ABF = CDF$ order is imp.

2. Imagine, everything defined

Distributive property

$A(B+C) = AB + AC$

$\neq BA + CA$ No → order is imp.

$(B+C)A = BA + CA$

$\neq AB + AC$ NO → unless they commute

3. Associative property

a) $A+B+C = (A+B)+C = A+(B+C)$

b) $ABC = (AB)C = A(BC)$

$\neq (AC)B$ NO only AC is same as BC

$\neq C(AB)$ NO I can't flip

Independent

$$\overset{Q_1}{\overset{1\,2\,3\,4}{}}\quad \overset{Q_2}{\overset{1\,2\,3\,4}{}}\quad \overset{Q_3}{\overset{1\,2\,3\,4}{}}$$

Q: Are $(-1,0,1,2)$, $(0,0,1,5,\cancel{0})$, $(1,0,0,10)$ independent in $\mathbb{R}^4$?

Why $\mathbb{R}^4 \rightarrow$ they live in $\mathbb{R}^4$, if it was $\mathbb{R}^3$ then no we have 4 co-ord.

Yes

* independent: Non of the given points is a linear combination of the other given points.

$Q_1 = (-1,0,1,2) = \alpha_1(0,0,1,5) + \alpha_2(1,0,0,10)$

we can't write them as a linear combination

$Q_2 = (0,1,5,0) = \alpha_1(-1,0,1,2) + \alpha_2(1,0,0,10)$

No such values for α_1 and α_2

$Q_3 = \alpha_1 Q_1 + \alpha_2 Q_2$

$Q_3 \neq$ a linear combination of Q_1 and Q_2

*Another meaning:

$\alpha_1(-1,0,1,2) + \alpha_2(0,1,5,0) + \alpha_3(1,0,0,10) = (0,0,0,0)$ always equal to the origion for $\mathbb{R}^4$

Solve for $\alpha_1, \alpha_2, \alpha_3$?

$\alpha_1 = \alpha_2 = \alpha_3$ is the only soln

$$\left.\begin{aligned} -\alpha_1 + 0\alpha_2 + \alpha_3 &= 0 \\ 0\alpha_1 + \alpha_2 + 0\alpha_3 &= 0 \\ 1\alpha_1 + 5\alpha_2 + 0\alpha_3 &= 0 \\ 2\alpha_1 + 0\alpha_2 + 10\alpha_3 &= 0 \end{aligned}\right\} \text{homogenous, 1 soln } (0,0,0,0) \text{ Independent}$$

If it was given $\alpha_1 = \alpha_2 = \alpha_3 = 0 \rightarrow$ independent.

Given independent, then I know the soln of homogenous

$Q_1 = (1,0,-1)$
$Q_2 = (-1,0,1)$
$Q_3 = (0,4,1)$
Q_1, Q_2, Q_3 are dependent
$Q_2 = -Q_1 + 0Q_3$ ✓
At least one point is a linear combination of the other given points.
Another meaning
$\alpha_1(1,0,-1) + \alpha_2(-1,0,1) + \alpha_3(0,4,1) = (0,0,0)$
There is a soln for $\alpha_1, \alpha_2, \alpha_3$ such that at least one of the α_i's $\neq 0$
→ The method is only to answer dependent or independent
→ Not augmented
→ We use row operations and kill only below the leaders
If there are 3 ~~lea~~ variables and I ended with 3 leaders then all independent
if I got 2 or 1 leaders then they are dependent.
5 points → 5 rows → 5 leaders

$$\begin{bmatrix} -1 & 0 & 1 & 2 \\ 0 & 1 & 5 & 0 \\ 1 & 0 & 0 & 10 \end{bmatrix} \xrightarrow{R_1+R_3 \to R_3} \begin{bmatrix} -1 & 0 & 1 & 2 \\ 0 & 1 & 5 & 0 \\ 0 & 0 & 1 & 12 \end{bmatrix}$$

3 leaders are independent
← We use this method to solve

3 points generated 3 leaders : yes, independent
If $Q_2 = -10Q_1 + Q_3$
given $\begin{bmatrix} Q_1 \\ Q_2 \\ Q_3 \end{bmatrix}$ ∴ $10Q_1 + Q_2 \to Q_2$, $-Q_3 + Q_2 \to Q_2$ } Substitute for Q, $10Q - Q_3 + Q_2$
Sub here and I'll get zero
∴ Dependent.
given $\begin{bmatrix} Q_1 \\ Q_2 \\ Q_3 \end{bmatrix}$ are independent. Imagine Q_1, Q_2, Q_3 live in $\mathbb{R}^5$ solve for
$aQ_1 + bQ_2 + cQ_3 = (0,0,0,0,0)$ a, b, c?
$a = b = c = 0$

4 points to be independent → I should have 4 leaders
If I got 3 leaders → dependent { (0,0,0,0) at least one of a_i is not zero}

$$\begin{cases} x_3 - x_4 = 0 \\ x_1 - x_4 = 1 \\ x_2 - x_4 = 2 \\ x_1 + x_2 - x_3 = 4 \end{cases}$$

is (0,5,3,1) a soln? No not in the soln set

↳ Soln set = {(2,3,1,1)}

Q: Re-write the system in matrix form $\underline{\underline{Ax}} = \underline{\underline{B}}$

any system of linear eqns can be written in ↑ form

$$\begin{array}{cccc} x_1 & x_2 & x_3 & x_4 \end{array}$$
$$\begin{bmatrix} 0 & 0 & 1 & -1 \\ 1 & 0 & 0 & -1 \\ 0 & 1 & 0 & -1 \\ 1 & 1 & -1 & 0 \end{bmatrix} \begin{bmatrix} x_1 & 2 \\ x_2 & 3 \\ x_3 & 1 \\ x_4 & 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \\ 2 \\ 4 \end{bmatrix}$$

Co-efficient matrix of the variables 4x4 | soln set 4x1 | 4x1

4x1 ⇝ If $\begin{bmatrix} 0 \\ 5 \\ 3 \\ 1 \end{bmatrix}$ then different answer

$$2\begin{bmatrix} 0 \\ 1 \\ 0 \\ 1 \end{bmatrix} + 3\begin{bmatrix} 0 \\ 0 \\ 1 \\ 1 \end{bmatrix} + 1\begin{bmatrix} 1 \\ 0 \\ 0 \\ -1 \end{bmatrix} + 1\begin{bmatrix} -1 \\ -1 \\ -1 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \\ 2 \\ 4 \end{bmatrix}$$

I got the eqns.

$$x_1\begin{bmatrix} 0 \\ 1 \\ 0 \\ 1 \end{bmatrix} + x_2\begin{bmatrix} 0 \\ 0 \\ 1 \\ 1 \end{bmatrix} + x_3\begin{bmatrix} 1 \\ 0 \\ 0 \\ -1 \end{bmatrix} + x_4\begin{bmatrix} -1 \\ -1 \\ -1 \\ 0 \end{bmatrix} = \begin{bmatrix} x_3 - x_4 \\ x_1 - x_4 \\ x_2 - x_4 \\ x_1 + x_2 - x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \\ 2 \\ 4 \end{bmatrix}$$

Know: A system of linear eqns in the form $AX = B$ is consistent (has a soln) if and only if B is a linear combination of $\boxed{\text{columns}}$ of A

$\underset{A}{[\ \ ]}\underset{X}{[\ ]} = \underset{B}{[\ ]}$ ← not A but only columns → linear combination

co-eff matrix → all variables; constant

A system of linear eqns is inconsistent if and only if B cannot be written as a linear combination of $\boxed{\text{columns}}$ of A

No soln set

↳ comb of A means 5[A], so I have to specify columns

Q : $Ax = B$
3×4 4×1 2nd column of A
3×1

1) Convince me that the system has ∞ many solns
Ⓐ Show it is consistant
Ⓑ Look for free variables

$$0\begin{bmatrix} \\ \\ \end{bmatrix} + 1\begin{bmatrix} \\ \\ \end{bmatrix} + 0\begin{bmatrix} \\ \\ \end{bmatrix} + 0\begin{bmatrix} \\ \\ \end{bmatrix} = \overset{B}{\begin{bmatrix} \\ \\ \end{bmatrix}} \rightarrow \text{2nd column of A}$$

∴ (0,1,0,0) is a soln ⇒ extra soln (0,2,0,0)

Rows
3x4 → 3 eqns and 4 unknowns, max # of leading is 3 (from # of eqns)
Which means that we have a free variable.
Consistent + Free variables ⇒ ∞ many solns.
If they wanted the 1st column, same answer.

Q : $x_3 - x_4 = 0$
$x_1 - x_4 = 1$
$x_2 - x_4 = 2$
$x_1 + x_2 - x_3 = 4$
Solnset = {(2,3,1,1)}

Not all variables to the power 1

Find the soln set for the following (steady) nonlinear system of eqns.
Eqn: $y_3^3 - y_4^2 = 0$
$y_1^2 - y_4^2 = 1$
$y_2^3 - y_4^2 = 2$
$y_1^2 + y_2^3 = 4$

4 variables y_1, y_2, y_3, y_4 and only y_4^2 is to the power 2 ⇝ y_4 is only squared
$x_3 = y_3^3$, $x_4 = y_4^2$, $x_1 = y_1^2$, $x_2 = y_2^3$
Read the substitutions, they are same as the eqns
(2,3,1,1)
$2 = y_1^2 \rightarrow y_1 = \pm\sqrt{2}$
$3 = y_2^3 \rightarrow y_2 = \sqrt[3]{3}$
$1 = y_3^3 \rightarrow y_3 = \sqrt[3]{1}$
$1 = y_4^2 \rightarrow y_4 = \pm 1$

Steady we change it into linear
They intersect in 4 points

Soln Set = $\{(\sqrt{2}, \sqrt[3]{3}, 1, 1), (\sqrt{2}, \sqrt[3]{3}, 1, -1), (-\sqrt{2}, \sqrt[3]{3}, 1, 1), (-\sqrt{2}, \sqrt[3]{3}, 1, 1)\}$

Q: $A = \begin{bmatrix} 3 & 1 & 2 & 1 \\ 0 & -1 & 1 & 1 \\ 2 & 1 & 3 & 4 \end{bmatrix}$ 3x4

① Find A^T (A transpose)

$A^T = \begin{bmatrix} 3 & 0 & 2 \\ 1 & -1 & 1 \\ 2 & 1 & 3 \\ 1 & 1 & 4 \end{bmatrix}$ 4x3

*1st row of A will be 1st column of A^T and so on.
*Flip each row to a column.

If A $n \times m$, A^T will be $m \times n$.

Q: $A = \begin{bmatrix} 1 & 0 & 3 \\ 0 & 2 & 4 \\ 3 & 4 & 10 \end{bmatrix} \rightsquigarrow 3\times3$ $\quad A^T = \begin{bmatrix} 1 & 0 & 3 \\ 0 & 2 & 4 \\ 3 & 4 & 10 \end{bmatrix} = A$

Definition: A, $n\times n$ is symmetric if $A^T = A$

*make sense, this apply only if the matrix is square

$W = \begin{bmatrix} 0 & -3 & 2 \\ 3 & 0 & 6 \\ -2 & -6 & 0 \end{bmatrix} \rightsquigarrow 3\times3$ $\quad W^T = \begin{bmatrix} 0 & 3 & -2 \\ -3 & 0 & -6 \\ 2 & 6 & 0 \end{bmatrix} = -W$

#rows = #columns

just the sign

Definition: A, $n\times n$ is skew-symmetric if $A^T = -A$

Result: (Know it):

A, $n\times n$. Then A = Symmetric matrix + skew symmetric

$$= \frac{1}{2}(A + A^T) + \frac{1}{2}(A - A^T)$$

we should have $\frac{1}{2}$ to get A

Know: A, B matrices

1) $(A \pm B)^T = A^T \pm B^T$

scaller 2) $(\alpha A)^T = \alpha A^T$

3) $(AB)^T = B^T A^T$ $\quad (WL)^T = L^T W^T$

4) $(A^T)^T = A$

Result: A, $n\times n$
$\alpha \in \mathbb{R}$

$\alpha(A + A^T)$ is symmetric

$\hookrightarrow [\alpha(A + A^T)]^T = \alpha(A + A^T)^T = \alpha(A^T + A) \rightsquigarrow$ Nothing change.

Result: A, $n\times n$
$\alpha \in \mathbb{R}$

$\alpha(A - A^T)$ is skewline

$[\alpha(A - A^T)]^T = \alpha(A^T - A) = -\alpha(A - A^T)$

Q: $\begin{bmatrix} 1 & 0 & 2 \\ 4 & 1 & 5 \\ -2 & 10 & 20 \end{bmatrix} = A$

Write A = symmetric + skew symmetric

Symmetric $= \frac{1}{2}(A + A^T) = \frac{1}{2}\begin{bmatrix} 2 & 4 & 0 \\ 4 & 2 & 15 \\ 0 & 15 & 40 \end{bmatrix} = \begin{bmatrix} 1 & 2 & 0 \\ 2 & 1 & 15/2 \\ 0 & 15/2 & 20 \end{bmatrix}$ L

Skew symmetric $= \frac{1}{2}(A - A^T)$

$A^T = \begin{bmatrix} 1 & 4 & -2 \\ 0 & 1 & 10 \\ 2 & 5 & 20 \end{bmatrix}$ $= \frac{1}{2}\begin{bmatrix} 0 & -4 & 4 \\ 4 & 0 & -5 \\ -4 & 5 & 0 \end{bmatrix} = \begin{bmatrix} 0 & -2 & 2 \\ 2 & 0 & -5/2 \\ -2 & 5/2 & 0 \end{bmatrix}$ W

$A = L + W$

$3 \times \boxed{} = 1$ → multiplicative identity (behaves in this way when multiplied)

multiplicative inverse

2 unique #s: 0 additive idendity

1 multiplication identity

$\begin{bmatrix} 2\times 2 \end{bmatrix} \begin{bmatrix} \end{bmatrix} = \{?\}$

matrix another matrix Identity Matrix

I_2 ⇝ 2x2 identity matrix $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

Whenever multiplication is allowed, whenever I multiply by I_2 nothing change.

Suppose A is 2x7 then $\underset{2\times 7}{A}\,\underset{7\times 7}{I_7} = \underset{\text{same}}{A}$ but $\underset{7\times 7}{I_7}\,\underset{2\times 7}{A}$ = undefined

$I_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ main diagonal in a square matrix

$I_2 \begin{bmatrix} 3 \\ 4 \end{bmatrix} = \begin{bmatrix} 3 \\ 4 \end{bmatrix}$

$I_n = \begin{bmatrix} \text{1 on main diagonal} & 0 \\ 0 & \end{bmatrix}$

$I_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ 3x3 identity matrix that behaves like 1

If multiplication is allowed, the $\underbrace{I_n}_{n\times n} \underbrace{A}_{\substack{\text{any matrix} \\ n\times k}} = A$

swap

Q: Find A^{-1} if possible

$$A = \begin{bmatrix} 0 & 1 & 2 \\ -1 & -1 & -1 \\ 0 & -1 & -1 \end{bmatrix}$$

* A^{-1} → We read it as A inverse
* $A^{-1} \neq 1/A$ → Cannot do that
* $AA^{-1} = I = A^{-1}A$ → Commute, the only case
* A^{-1} makes sence only if A is a square nxn matrix

by row operations we get one another
A and I_3 are equivilant

Answer:

$$\left[\begin{array}{ccc|ccc} 0 & 1 & 2 & 1 & 0 & 0 \\ -1 & -1 & -1 & 0 & 1 & 0 \\ 0 & -1 & -1 & 0 & 0 & 1 \end{array}\right]$$

Do row operations as taught on the whole matrix, and stop when I reach the following

$$\left[\begin{array}{c|c} \text{If } I_3 & A^{-1} \\ \text{If can't get } I_3 \text{ here, then} & A^{-1} \text{ DNE} \end{array}\right]$$

$R_1 + R_2 \to R_2$
$R_1 + R_3 \to R_3$

$$\left[\begin{array}{ccc|ccc} 0 & 1 & 2 & 1 & 0 & 0 \\ -1 & 0 & 1 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 & 0 & 1 \end{array}\right] \xrightarrow{-R_2} \left[\begin{array}{ccc|ccc} 0 & 1 & 2 & 1 & 0 & 0 \\ 1 & 0 & -1 & -1 & -1 & 0 \\ 0 & 0 & 1 & 1 & 0 & 1 \end{array}\right]$$

if zero, I don't cont.: (NO LEADER, NO INVERSE)

$2R_3 + R_1 \to R_1$
$R_3 + R_2 \to R_2$

$$\left[\begin{array}{ccc|ccc} 0 & 1 & 0 & -1 & 0 & -2 \\ 1 & 0 & 0 & 0 & -1 & 1 \\ 0 & 0 & 1 & 1 & 0 & 1 \end{array}\right]$$

leader only from this side.

Store

By interchanging rows

$$\left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 0 & -1 & 1 \\ 0 & 1 & 0 & -1 & 0 & -2 \\ 0 & 0 & 1 & 1 & 0 & 1 \end{array}\right]$$

I_3 | A^{-1}

$$A^{-1} = \begin{bmatrix} 0 & -1 & 1 \\ -1 & 0 & -2 \\ 1 & 0 & 1 \end{bmatrix}$$

Know:

A nxn will have A^{-1} iff each row has a leader

Q: Find A^{-1} if possible

$$A = \begin{bmatrix} 1 & -1 \\ -2 & 2 \end{bmatrix}$$

$$\left[\begin{array}{cc|cc} 1 & -1 & 1 & 0 \\ -2 & 2 & 0 & 1 \end{array}\right] \xrightarrow{2R_1 + R_2 \to R_2} \left[\begin{array}{cc|cc} 1 & -1 & 1 & 0 \\ 0 & 0 & 2 & 1 \end{array}\right]$$

NO

A^{-1} DNE.

Let A as in the previous question, Find the solution set to the system

$$A \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 3 \\ 0 \\ -2 \end{bmatrix}$$

$$\begin{bmatrix} 0 & 1 & 2 \\ -1 & -1 & -1 \\ 0 & -1 & -1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 3 \\ 0 \\ -2 \end{bmatrix}$$

$$A^{-1}AX = A^{-1} \begin{bmatrix} 3 \\ 0 \\ -2 \end{bmatrix}$$

$$I_3 X = A^{-1} \begin{bmatrix} 3 \\ 0 \\ -2 \end{bmatrix} \rightarrow X = \begin{bmatrix} 0 & -1 & 1 \\ -1 & 0 & -2 \\ 1 & 0 & 1 \end{bmatrix} \begin{bmatrix} 3 \\ 0 \\ -2 \end{bmatrix}$$

$$\cancel{X = \begin{bmatrix} 0 & 1 & 2 \\ -1 & -1 & -1 \\ 0 & -1 & -1 \end{bmatrix}}$$

$$= 3 \begin{bmatrix} 0 \\ -1 \\ 1 \end{bmatrix} + 0 \begin{bmatrix} -1 \\ 0 \\ 0 \end{bmatrix} - 2 \begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix} = \begin{bmatrix} -2 \\ 1 \\ 1 \end{bmatrix} \quad \begin{matrix} x_1 = -2 \\ x_2 = 1 \\ x_3 = 1 \end{matrix}$$

$$\{(-2, 1, 1)\}$$

A result for only 2x2

$A_{2\times2}$, $\begin{bmatrix} a & c \\ b & d \end{bmatrix}$ A^{-1} exist iff $D = ad - cb \neq 0$

$\therefore A^{-1} = \frac{1}{D}\begin{bmatrix} d & -c \\ -b & a \end{bmatrix}$

Convince me why is it true! $AA^{-1} = I_2$

Q) $A = \begin{bmatrix} 3 & -2 \\ 7 & -2 \end{bmatrix}$ What is A^{-1} ?

1) $D = 3\times-2-(7\times-2) = 8 \neq 0$

2) $A^{-1} = \frac{1}{8}\begin{bmatrix} -2 & 2 \\ -7 & 3 \end{bmatrix} =$

Q) Given $A^{-1} = \begin{bmatrix} 3 & 0 & 2 \\ 3 & 1 & 4 \\ -3 & 0 & -1 \end{bmatrix}$

$\boxed{a}$ Find A

$\boxed{\text{RULE 8 } (A^{-1})^{-1} = A}$

$\left[\begin{array}{ccc|ccc} 3 & 0 & 2 & 1 & 0 & 0 \\ 3 & 1 & 4 & 0 & 1 & 0 \\ -3 & 0 & -1 & 0 & 0 & 1 \end{array}\right] \sim$ Row operations $\left[\begin{array}{c|c} I_3 & A \end{array}\right]$

$\boxed{b}$ Find $(A^T)^{-1}$

$\boxed{\text{RULE 8 } (A^T)^{-1} = (A^{-1})^T}$ → Why is it true?

$AA^{-1} = I_n$

$(AA^{-1})^T = (I_n)^T$ ↳ remains the same

$(A^{-1})^T A^T = I_n$ $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

↳ what matrix I multiply by to get the identity

$(A^{-1})^T = (A^T)^{-1}$

$\begin{bmatrix} 3 & 3 & -3 \\ 0 & 1 & 0 \\ 2 & 4 & -1 \end{bmatrix}$

$A^T = ((A^{-1})^T)^{-1}$

$\boxed{c}$ Find $(4A)^{-1}$

$\boxed{\text{RULE 8 } (\alpha A)^{-1} = \frac{1}{\alpha} A^{-1}, \alpha \neq 0}$ → why is it true?

$4A \times \boxed{(4A)^{-1}} = I_3$

$\frac{1}{4} A^{-1}$

$(A^{-1})^T = (A^T)^{-1}$

$= \frac{1}{4} A^{-1}$

Determinant

A 2x2, $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$

$|A| = \det(A)$ → both mean determinent, should be nxn square

$= ad - bc$

Q) $A = \begin{bmatrix} 0 & 2 & 4 \\ 5 & 2 & 3 \\ 0 & 10 & 2 \end{bmatrix}$ Find $|A|$

Choose any row or column of A (recommended to choose the one with more zeros).

1st column:

$0(-1)^{1+1} \begin{vmatrix} 2 & 3 \\ 10 & 2 \end{vmatrix} + 5(-1)^{2+1} \begin{vmatrix} 2 & 4 \\ 10 & 2 \end{vmatrix} + 0(-1)^{3+1} \begin{vmatrix} 2 & 4 \\ 2 & 3 \end{vmatrix} = -5 \begin{vmatrix} 2 & 4 \\ 10 & 2 \end{vmatrix}$

$= -5(4-40) = -5 \times -36 = 180$

Extra: 2nd row:

$5(-1)^{2+1} \begin{vmatrix} 2 & 4 \\ 10 & 2 \end{vmatrix} + 2(-1)^{2+2} \begin{vmatrix} 0 & 4 \\ 0 & 2 \end{vmatrix} + 3(-1)^{2+3} \begin{vmatrix} 0 & 2 \\ 0 & 10 \end{vmatrix}$

$= -5(4-40) = -5 \times -36 = 180$

Def 8

① $\begin{bmatrix} 1 & 2 & 3 & 4 \\ 0 & 5 & 1 & 0 \\ 0 & 0 & 2 & 1 \\ 0 & 0 & 0 & 10 \end{bmatrix}$ All #'s under the main diagonal are zero's

Upper triangular

$|A| = 1 \times 5 \times 2 \times 10 = 100$

② $\begin{bmatrix} 2 & 0 & 0 & 0 & 0 \\ 3 & 1 & 0 & 0 & 0 \\ 0 & 0 & 5 & 0 & 0 \\ 2 & 4 & 1 & 0 & 0 \\ 0 & 0 & 1 & 5 & 10 \end{bmatrix}$ All #'s above the main diagonal are zero's

Lower triangular

$|A| = 2 \times 1 \times 5 \times 0 \times 10 = 200$

③ $\begin{bmatrix} 3 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$ Diagonal matrix

$|A| = 3 \times 0 \times 0 = 0$

* We say (A) is a triangular matrix if (A) is in the form 1, or 2, or 3.
* Know: A square nxn triangular matrix

$|A|$ = multiply all #'s on the main diagonal

Q. Find $|A|$ using the definition

$$A = \begin{bmatrix} 2 & 4 & 6 & 8 \\ -2 & -3 & -4 & 1 \\ -6 & -12 & 13 & 2 \\ -4 & -8 & -12 & 18 \end{bmatrix}$$

Column 1

$$2(-1)^{1+1}\begin{vmatrix} -3 & -4 & 1 \\ -12 & 13 & 2 \\ -8 & -12 & 18 \end{vmatrix} + -2(-1)^{2+1}\begin{vmatrix} 4 & 6 & 8 \\ -12 & 13 & 2 \\ -8 & -12 & 18 \end{vmatrix} + -6(-1)^{3+1}\begin{vmatrix} 4 & 6 & 8 \\ -3 & -4 & 1 \\ -8 & -12 & 18 \end{vmatrix} + -4(-1)^{4+1}\begin{vmatrix} 4 & 6 & 8 \\ -3 & -4 & 1 \\ -12 & 13 & 2 \end{vmatrix}$$

Idea use as you wish to change A to upper triangular

these row operations have no effect on the det.

$$A = \begin{bmatrix} 2 & 4 & 6 & 8 \\ -2 & -3 & -4 & 1 \\ -6 & -12 & 13 & 2 \\ -4 & -8 & -12 & 18 \end{bmatrix} \xrightarrow{\frac{1}{2}R_1} \begin{bmatrix} 1 & 2 & 3 & 4 \\ -2 & -3 & -4 & 1 \\ -6 & -12 & 13 & 2 \\ -4 & -8 & -12 & 18 \end{bmatrix} \xrightarrow[4R_1+R_4 \to R_4]{2R_1+R_2 \to R_2,\ 6R_1+R_3 \to R_3} \begin{bmatrix} 1 & 2 & 3 & 4 \\ 0 & 1 & 2 & 9 \\ 0 & 0 & 31 & 26 \\ 0 & 0 & 0 & 32 \end{bmatrix}$$

$|A| \qquad \frac{1}{2}|A| \qquad |F|$

$\frac{1}{2}R_1 \cdot 5R_2, 3R_3 \to \frac{1}{2}\times 5 \times 3|A|$

$|F| = 1 \times 1 \times 31 \times 32 = 992$

$|F| = \frac{1}{2}|A| \to |A| = 2|F| = 2 \times 992 = 1984$

The effect of row operations on $|A|$

1) $A_{n\times n} \underset{\alpha=0}{\xrightarrow{\alpha R_i}} B$, $|B| = \alpha|A|$

2) $A \xrightarrow{R_i \leftrightarrow R_k} B$, $|B| = -|A|$ $\quad R_1 \leftrightarrow R_3,\ R_3 \leftrightarrow R_4,\ R_1 \leftrightarrow R_2 \quad |B| = -|A|$

3) $A \xrightarrow{\alpha R_1 + R_k} B$, $|B| = |A|$

We can change the matrix to lower triangle and kill above

Q. $A \xrightarrow{3R_1,\ -\frac{2}{3}R_3} B \xrightarrow{R_1 \leftrightarrow R_3} C \xrightarrow[-5R_3+R_4 \to R_4]{-2R_3+R_1 \to R_1,\ 4R_3+R_2 \to R_2} \begin{bmatrix} 0 & 3 & 2 & 4 \\ 2 & 2 & 1 & 4 \\ 0 & 0 & 0 & 10 \\ 0 & 0 & 3 & 12 \end{bmatrix} F \xrightarrow{R_1 \leftrightarrow R_2,\ R_3 \leftrightarrow R_4} \begin{bmatrix} 2 & 2 & 1 & 4 \\ 0 & 3 & 2 & 4 \\ 0 & 0 & 3 & 12 \\ 0 & 0 & 0 & 10 \end{bmatrix} M$

4×4

$|A| \qquad 3 \times -\frac{2}{3}|A| = -2|A| \qquad 2|A| \qquad 2|A| \qquad 2|A|$

stare + switch

Find $|B| = -180$

Find $|C| = 2|A| = 180$

Find $|F| = 2|A| = 2 \times 90 = 180$

Find $|A| = 90$

$|M| = 2 \times 3 \times 3 \times 10 = 180$

$|M| = 2|A| \to |A| = 90$

Q. $\begin{vmatrix} 1 & 2 & 3 & 10 \\ -1 & 5 & 2 & 1 \\ 0 & 0 & 0 & 13 \\ -2 & -4 & 12 & 1 \end{vmatrix}$ → Two vertical lines means det.
use row operations

$R_1+R_2 \to R_2$, $2R_1+R_4 \to R_4$ $\begin{vmatrix} 1 & 2 & 3 & 10 \\ 0 & 7 & 5 & 11 \\ 0 & 0 & 0 & 13 \\ 0 & 0 & 18 & 21 \end{vmatrix}$ $R_3 \leftrightarrow R_4$ interchange $\begin{vmatrix} 1 & 2 & 3 & 10 \\ 0 & 7 & 5 & 11 \\ 0 & 0 & 18 & 21 \\ 0 & 0 & 0 & 13 \end{vmatrix}$ → $-|A| = 1638$

$|A| = -1638$

Q. $\underset{3\times3}{A} \xrightarrow[-\frac{1}{2}R_3]{2R_1} A_1 \xrightarrow{-3R_2+R_3 \to R_3} A_2 \xrightarrow{R_3 \leftrightarrow R_2} \begin{bmatrix} 2 & 1 & 0 \\ 0 & 3 & 1 \\ 0 & 0 & 4 \end{bmatrix}$

$2 \times -\frac{1}{2}|A| = -|A|$ $\quad -|A|$ $\quad |A| = 24$

① Find $|A| = 24$

② Find A → Go backward

$A \to \begin{bmatrix} 1 & 1/2 & 0 \\ 0 & 0 & 4 \\ 0 & -6 & -26 \end{bmatrix} \xleftarrow[-2R_3]{1/2 R_1} \begin{bmatrix} 2 & 1 & 0 \\ 0 & 0 & 4 \\ 0 & 3 & 13 \end{bmatrix} \xleftarrow{3R_2+R_3 \to R_3} \begin{bmatrix} 2 & 1 & 0 \\ 0 & 0 & 4 \\ 0 & 3 & 1 \end{bmatrix} \xleftarrow{R_3 \leftrightarrow R_2}$

$2R_1$ is $\frac{1}{2}R_1$, $-\frac{1}{2}R_3$ is $-2R_3$

was $-3R_2+R_3 \to R_3$ then $3R_2+R_3 \to R_3$

$4 \times 3 + 1 = 13$

* If I have a row with zeros only → det = zero

Properties of determinents:

A nxn, B nxn

1) $|\alpha A| = \alpha^n |A|$ → multiply 1 row by α then multiply each row by α

→ α is a fixed real # $\alpha = 2, n = 5, 2^5$

2) $|AB| = |A||B|$

det of multiplication = multiplication of the det

3) $|A \pm B|$ need not $= |A| \pm |B|$

$|A \pm B| \neq |A| \pm |B|$

→ Proof: $A = \begin{bmatrix} 1 & 3 \\ 0 & 0 \end{bmatrix}$, $B = \begin{bmatrix} 0 & 0 \\ 0 & 5 \end{bmatrix}$, $A+B = \begin{bmatrix} 1 & 3 \\ 0 & 5 \end{bmatrix}$

$|A| = 0$ $\quad |B| = 0$

$\left.\begin{array}{l} |A+B| = 5 \\ |A|+|B| = 0 \end{array}\right\}$ $5 \neq 0$ need not be true. I can't seperate

4) $|A| = |A^T|$ 2nd row $|A|$ is 2 column $|A^T|$

5) Assume A^{-1} exist $|A^{-1}| = 1/|A|$

$\left.\begin{array}{l} A^{-1}A = I_n \\ |A^{-1}A| = |I_n| \end{array}\right\}$ diagonal matrix = 1, det of $I_n = 1$ → result

from point 2 → $|A^{-1}||A| = 1$ → $|A^{-1}| = 1/|A|$

A $n \times n$

$\left[A \mid I_n\right] \sim \left[I_n \mid A^{-1}\right]$ → When I want to find the inverse I should get this matrix

but if ↓ → If I didn't get I_n, then no inverse

This mean I have free variables $|A|=0$ ← $\left[\begin{array}{c|c} \text{No } I_n & A^{-1} \text{ DNE} \end{array}\right]$

↳ Assume a consitant sys. ↳ must have a zero row

$|A|=0$ → ∞ many soln
→ No soln

determinent $= 0$

$\alpha|A| = 0$

then $|A| = 0$

Result:

$n \times n$ A

① A is singular (non invertible), A^{-1} DNE iff $|A|=0$

② A is a nonsingular (invertible), iff A^{-1} then $|A| \neq 0$

Q.

A, $n \times n$, Find the soln for $AX=b$

$[A \mid b] \xrightarrow{\text{row op.}} [I_n \mid b']$

Result:

(Know) A, $n \times n$,

The system of linear eqn $AX=b$ has $\boxed{\text{unique}}$ soln iff A is $\boxed{\text{nonsingular}}$ (invertible) iff $\boxed{|A| \neq 0}$. If one is true, then the other two ~~is~~ are true.

Q.

$|A|=3$. Is the system soln unique if $AX = \begin{bmatrix} 12 \\ 3 \\ 1 \end{bmatrix}$? yes

Result:

(Know) $AX=b$, System of linear eqn assume $|A|=0$ either no soln or ∞ many soln.

Q1) Find the soln set

$3X_1 - 2X_2 + X_3 = 0$
$X_1 + X_2 - X_3 = 0$
$4X_1 - 2X_2 + 3X_3 = 0$

homogeneous → consistent
$(0,0,..)$ slon.

Q2) $A = \begin{bmatrix} 3 & -2 & 1 \\ 1 & 1 & -1 \\ 4 & -2 & 3 \end{bmatrix}$ find the zeros of A = Nul of A

$Z(A)$ same $N(A)$

Z(A) is a soln to the homogeneous system $AX = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$

Q1 = Q2 same question, but different form

A is 10x10, $|A| = 2017$, find zeros of A
$|A| \neq 0$ ⇝ unique ⇝ Z(A) soln set has one point ⇝ $Z(A) = \{(0,0,\ldots)\}$

$\underset{2\times3}{A^3} = \underset{2\times3}{A}\,\underset{2\times3}{A}\,\underset{2\times3}{A}$ ⇝ won't work (size)

Know: A^m means $\underbrace{AAA\ldots A}$ → m times, assume A is a square matrix (nxn) and m is a +ve integer

$A^{-3} = A^{-1}A^{-1}A^{-1}$

Know: $A^{-m} = \underbrace{A^{-1}A^{-1}\ldots A^{-1}}$ → m times, assuming A is a nonsingular and square

Find the inverse to A^7 A^{-7}
Find the inverse to A^2 A^{-2}

$A^2A^{-2} = AAA^{-1}A^{-1} = AI_nA^{-1} = AA^{-1} = I_n$

$|A| = 4$, $|A^3| = |AAA| = |A||A||A| = 4\times4\times4 = 4^3$

$(3A)^{-1} = \frac{1}{3}A^{-1}$

Know: $(\alpha A)^{-1} = \frac{1}{\alpha}A^{-1}$
$\alpha \neq 0$

$(AB)^T = B^TA^T$

Know: $(AB)^{-1} = B^{-1}A^{-1}$

$ABB^{-1}A^{-1} = AI_nA^{-1}$, $AA^{-1} = I_n$

$\therefore (AB)^{-1} = B^{-1}A^{-1}$

Multiplication → Linear combination ⇝ Find columns of C
↳ Dot product ⇝ calculate $c_{i,k}$
↳ I want a specific point in C

$$\underset{n\times m}{A}\ \underset{m\times k}{B} = \underset{n\times k}{C}$$

$$\begin{bmatrix} c_{11} & c_{1,2} & \cdots & c_{1,k} \\ c_{2,1} & c_{2,2} & \cdots & c_{2,k} \\ c_{3,k} & & & \\ \vdots & & & \\ c_{n,1} & \cdots & \cdots & c_{n,k} \end{bmatrix}$$

Q: $\underset{3\times 4}{A} = \begin{bmatrix} 1 & 2 & 4 & 1 \\ -1 & 0 & 2 & 4 \\ 3 & 1 & 4 & 5 \end{bmatrix}$ $\underset{4\times 3}{B} = \begin{bmatrix} 2 & 0 & 1 \\ 3 & 3 & 4 \\ 2 & 1 & 6 \\ -1 & 4 & 10 \end{bmatrix}$ $AB = C$

$c_{2,3}$ → 2nd row (A), 3rd column (B)

$$\begin{bmatrix} -1 & 0 & -2 & 4 \end{bmatrix} \cdot \begin{bmatrix} 1 \\ 4 \\ 6 \\ 10 \end{bmatrix} = -1 \times + 0\times 4 + 12 + 40 = 51$$

$c_{3,1}$ → 3rd row (A), 1st column (B)

$$\begin{bmatrix} 3 & 1 & 4 & 5 \end{bmatrix} \cdot \begin{bmatrix} 2 \\ 3 \\ 2 \\ -1 \end{bmatrix} = 3\times 2 + 3\times 1 + 4\times 2 + 5\times -1 = 12$$

Eigen values and eigen vectors

Assume A is 4x4

Assume $Q = (1,3,6,2) \in \mathbb{R}^4$ {non zero}

Assume I told you $AQ^T = 3Q^T$ ⇝ $A\begin{bmatrix} 1 \\ 3 \\ 6 \\ 2 \end{bmatrix} = 3\begin{bmatrix} 1 \\ 3 \\ 6 \\ 2 \end{bmatrix}$

I found the transpose 4x4 and 4x1 →

→ 3 is eigen value

↳ Q is an eigen vector → eigen point

$A\begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} = 5\begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$ ⇝ If (0,0,0,0) is the only point where $A\begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} = 5\begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$ we don't call 5 an eigen value, we don't call $\begin{bmatrix} 0 \\ 0 \end{bmatrix}$ an eigen point

$\begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$ $\begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$

Q: A 7×7, assume I told you 0.5 is an eigen value of A what does that mean?

Answer: There is at least 1 point that lives in R^7, say Q such that $Q \neq (0,0,\dots 0)$ and $AQ^T = 0.5\,Q^T$

Q: A 4×4, $\frac{2}{3}$ is an eigen value in A?

Answer: There is at least 1 point that lives in R^4, say Q such that $Q \neq (0,0\dots 0)$ and $AQ^T = \frac{2}{3}Q^T$

Q: 3 is an eigen value of A 5×5?

Answer: There is at least on point, say Q, that lives in R^5 such that $Q \neq (0,0,0,0,0)$ and $AQ^T = 3Q^T$

$Q = (a_1, a_2, a_3, a_4, a_5)$, $A\begin{bmatrix} a_1 \\ a_2 \\ a_3 \\ a_4 \\ a_5 \end{bmatrix} = 3\begin{bmatrix} a_1 \\ a_2 \\ a_3 \\ a_4 \\ a_5 \end{bmatrix}$ → 3 eigen value, Q eigenpoint (vector)

$A\begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} = 3\begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$ ⇝ True, but I call it eigen because there is at least one point other than this point.

Q: 5 is not an eigen value of A, 4×4?

Answer: $(0,0,0,0)$ is the only point in R^4 that satisfies

$A\begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} = 5\begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$ → No other points

Q: $A = \begin{bmatrix} -1 & 2 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{bmatrix}$

(a) Find all eigen values of A

(b) For each eigen value α of A Find E_α = Set of all points in R, say Q S.T. $AQ^T = \alpha Q^T$

Assume α is an eigen value of A.

There is a point $Q \neq (0,0,0)$ S.T. $AQ^T = \alpha Q^T$

$\alpha Q^T - AQ^T = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$ (↳ cuz 3x3) ⇝ $(\underbrace{\alpha}_{\#} - \underbrace{A}_{\text{matrix}})Q^T = X$ (such that) I can't subtract # from matrix

$\underbrace{\alpha I_3 Q^T}_{\text{same as } \alpha Q^T} - AQ^T = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$ ⇝ $(\alpha I_3 - A)Q^T = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$ ↳ now I defined and I can subtract ⇝ $Z(A), N(A)$ ↳ zeros of A

Q belongs to the $Z(\alpha I_3 - A) = N(\alpha I_3 - A)$ = solution set of the homogeneous system $(\alpha I_3 - A)\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$

→ det = 0, not unique

det ≠ 0

For (1):

$C_A(\alpha) = |\alpha I_{③} - A|$ ← cuz 3x3

↳ Charecteristic polynomial of A

To find α's set $C_A(\alpha) = |\alpha I_3 - A| = 0$, solve for α.

det = 0 → soln that are not zero.

$$C_A(\alpha) = \left| \begin{bmatrix} \alpha & 0 & 0 \\ 0 & \alpha & 0 \\ 0 & 0 & \alpha \end{bmatrix} - \begin{bmatrix} -1 & 2 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{bmatrix} \right|$$

$$= \begin{vmatrix} \alpha+1 & -2 & -1 \\ 0 & \alpha-1 & 0 \\ 0 & 0 & \alpha-2 \end{vmatrix} = (\alpha+1)(\alpha-1)(\alpha-2)$$

↳ Charecteristic polynomial

Find all Qs in R^3 when $AQ^T = \frac{1}{2} Q^T$?
$Q = 0$, $\frac{1}{2}$ is not an eigen value

eigen values set $(\alpha+1)(\alpha-1)(\alpha-2) = 0 \rightarrow \left.\begin{array}{l}\alpha = 1 \\ \alpha = -1 \\ \alpha = 2\end{array}\right\} \Rightarrow$ Eigen Values

Now we find $E_{-1} = Z(-1 I_3 - A)$

↳ eigen space, set of all points, R say Q S.T. $AQ^T = \alpha Q^T$

$$E_{-1} = Z \begin{bmatrix} 0 & -2 & -1 \\ 0 & 0 & 0 \\ 0 & 0 & -3 \end{bmatrix} = N \begin{bmatrix} 0 & -2 & -1 \\ 0 & 0 & 0 \\ 0 & 0 & -3 \end{bmatrix}$$

$$\left[\begin{array}{ccc|c} 0 & -2 & -1 & 0 \\ 0 & -2 & 0 & 0 \\ 0 & 0 & -3 & 0 \end{array}\right]$$ ↝ now we solve (row by row, leader, kill) OR directly

$-3X_3 = 0 \rightarrow \boxed{X_3 = 0}$ $\quad -2X_2 = 0 \rightarrow \boxed{X_2 = 0}$

$(0)X_1 - 2X_2 - X_3 = 0 \rightarrow 0X_1 - 2(0) - (0) = 0 \Rightarrow X_1 \in \mathbb{R}$ Free

X_2, X_3 leaders

$E_{-1} = \{(X_1, 0, 0) \mid X_1 \in \mathbb{R}\}$

$= \{X_1(1,0,0) \mid X_1 \in \mathbb{R}\} = \text{Span}\{(1,0,0)\}$

↳ Set of any real #, so α points

$A\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} = -1 \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$ ↝ $A\begin{bmatrix} 2 \\ 3 \\ 4 \end{bmatrix} = -1 \begin{bmatrix} 2 \\ 3 \\ 4 \end{bmatrix}$ Not in the spane wrong statement

$E_1 = Z(1I_3 - A) = Z\begin{bmatrix}2 & -2 & -1\\ 0 & 0 & 0\\ 0 & 0 & -1\end{bmatrix}$ $x_3 = 0 \to$ Leading

$2x_1 - 2x_2 - x_3 = 0$ ($x_3 \to$ zero)

$x_1 = x_2$ (x_1 Leading, x_2 Free)

$E_1 = \{(x_2, x_2, 0) \mid x_2 \in \mathbb{R}\} = \{x_2(1,1,0) \mid x_2 \in \mathbb{R}\} = span\{(1,1,0)\}$

$A\begin{bmatrix}1\\1\\0\end{bmatrix} = 1\begin{bmatrix}1\\1\\0\end{bmatrix}$

$E_2 = Z(2I_3 - A) = Z\begin{bmatrix}3 & -2 & -1\\ 0 & 1 & 0\\ 0 & 0 & 0\end{bmatrix}$ $x_1 = x_3$ (x_1 Leading, x_3 Free)

$x_2 = 0$

A, 8x8, 5 eigenvalue

E_5 ? Set of all points in $\mathbb{R}^8$, say Q, s.t. $AQ^T = 5Q^T$

Q: $A = \begin{bmatrix}0 & 0 & 0 & 0\\ 1 & 0 & 0 & 0\\ 0 & 1 & 0 & -1\\ 0 & 0 & 1 & -2\end{bmatrix}$ 4x4

① Find eigenvalues of A

② For each eigen value α find E_α

① Set $C_A(\alpha) = |\alpha I_4 - A| = 0$ $\alpha I_4 - A$

$$\left|\begin{bmatrix}\alpha & 0 & 0 & 0\\ 0 & \alpha & 0 & 0\\ 0 & 0 & \alpha & 0\\ 0 & 0 & 0 & \alpha\end{bmatrix} - \begin{bmatrix}0 & 0 & 0 & 0\\ 1 & 0 & 0 & 0\\ 0 & 1 & 0 & -1\\ 0 & 0 & 1 & -2\end{bmatrix}\right| = \left|\begin{bmatrix}\alpha & 0 & 0 & 0\\ -1 & \alpha & 0 & 0\\ 0 & -1 & \alpha & 1\\ 0 & 0 & -1 & \alpha+2\end{bmatrix}\right|$$

if not triangular row operations won't work.

Know: If we don't have triangular, questions with eigenvalues use definition of det

$= \alpha(-1)^{1+1}\begin{vmatrix}\alpha & 0 & 0\\ -1 & \alpha & 1\\ 0 & -1 & \alpha+2\end{vmatrix} = \alpha(-1)^{1+1}\left[\alpha(-1)\begin{vmatrix}\alpha & 1\\ -1 & \alpha+2\end{vmatrix}\right]$

$= \alpha\left(\alpha\begin{vmatrix}\alpha & 1\\ -1 & \alpha+2\end{vmatrix}\right)$

$= \alpha(\alpha\underbrace{(\alpha^2 + 2\alpha + 1)}_{(\alpha+1)(\alpha+1)}) = 0$

$*A\begin{bmatrix}0\\3\\6\\3\end{bmatrix} = 0\begin{bmatrix}0\\3\\6\\3\end{bmatrix} = \begin{bmatrix}0\\0\\0\\0\end{bmatrix}$

$\alpha=0$ repeated twice
$\alpha=-1$ repeated twice

$$E_0 = Z\begin{bmatrix} 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & -1 & 2 \\ 0 & -1 & 0 & 1 \end{bmatrix}$$

$X_1 = 0 \rightsquigarrow$ leader (X_1)
$-X_2 = -X_4 \to X_2 = X_4 \rightsquigarrow$ leader (X_2)
$X_3 = 2X_4 \rightsquigarrow$ leader (X_3)

$\{(0, X_4, 2X_4, X_4) \mid X_4 \in \mathbb{R}\}$
Span $= \{(0,1,2,1)\}$

$10\times10 \longrightarrow \alpha$ 10 #s $6\times6 \to \alpha$ 6 #s
$4\times4 \longrightarrow \alpha$ 4 #s

$$\left(\begin{bmatrix} 3 & 0 & 0 \\ 0 & 5 & 0 \\ 0 & 0 & 2 \end{bmatrix}\right)^4 = \begin{bmatrix} 3^4 & 0 & 0 \\ 0 & 5^4 & 0 \\ 0 & 0 & 2^4 \end{bmatrix}$$

When you multiply two diagonal multiply the diagonals only

$$\underbrace{\begin{bmatrix} 3 & 0 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & 10 \end{bmatrix}}_{\text{diagonal}}\underbrace{\begin{bmatrix} 4 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 20 \end{bmatrix}}_{\text{diagonal}} = \begin{bmatrix} 3\times4 & 0 & 0 \\ 0 & -2\times-1 & 0 \\ 0 & 0 & 10\times20 \end{bmatrix}$$

Know: When you multiply diagonal matrices we just multiply the #'s on the main diagonal

Q: $A = \begin{bmatrix} 2 & 0 & 3 \\ 0 & -5 & 1 \\ 0 & 0 & 4 \end{bmatrix}$ Is it diagonolizable? if yes, find an invertible (non-singular) matrix Q and a diagonal matrix D s.t. $QDQ^{-1} = A$

$QDQ^{-1} = A$ ← Can be written ← If A can be written in this way this means it is diagnolizable.

$Q^{-1}QDQ^{-1} = AQ^{-1}$

$I_n DQ^{-1} = AQ^{-1}$

$DQ^{-1} = AQ^{-1}$ ↪ multiply now to the right

$D = Q^{-1}AQ$

Set $C_A(\alpha) = 0$, find α

$$C_A(\alpha) = |\alpha I_3 - A| = \left| \begin{bmatrix} \alpha & 0 & 0 \\ 0 & \alpha & 0 \\ 0 & 0 & \alpha \end{bmatrix} - \begin{bmatrix} 2 & 0 & 3 \\ 0 & -5 & 1 \\ 0 & 0 & 4 \end{bmatrix} \right| = \left| \overbrace{\begin{bmatrix} \alpha-2 & 0 & -3 \\ 0 & \alpha-5 & -1 \\ 0 & 0 & \alpha-4 \end{bmatrix}}^{\alpha I_3 - A} \right|$$

we are lucky! diagonal

$(\alpha-2)(\alpha-5)(\alpha-4) = 0$

$\alpha = 2$ $\alpha = 4$ repeated one time.

$\alpha = 5$ They garantee the existance of nonzero points R^3

For each α Find E_α (eigenspace, eigenpoints)

$E_\alpha = Z(\alpha I_3 - A)$

$= N(\alpha I_3 - A)$

$\alpha = 2$

$E_2 = Z(2I_3 - A) = Z\begin{bmatrix} 0 & 0 & -3 \\ 0 & 7 & -1 \\ 0 & 0 & -2 \end{bmatrix}$

$-3X_3 = 0$ $\boxed{X_3 = 0}$ leader

$7X_2 - X_3 = 0$ $\boxed{X_2 = 0}$ leader

$-2X_3 = 0$

X_1 is Free.

$\{(X_1, 0, 0) \mid X_1 \in \mathbb{R}\} \rightarrow \text{span}\{(1,0,0)\}$ we can write $(5,0,0)$ it lives in R^3. multiply by 5

2 repeated 1 time ∴ 1 point, if 5 repeated 12 times ∴ span of 12 points

Big Know:

Assume A has eigen values $\alpha_1, \alpha_2, \alpha_3 \cdots \alpha_m$ assume α_1 repeated K_1 times, assume α_2 repeated K_2 times. A is diagonizable iff α_i repeated K_i times. E_{α_i} = span {K points}

Ex. 3 is repeated 2 times ∴ E_3 = Span {2 points}

$\alpha = -5$

$E_{-5} = Z(-5I_3 - A) = Z \begin{bmatrix} -7 & 0 & -3 \\ 0 & 0 & -1 \\ 0 & 0 & -9 \end{bmatrix}$ $X_3 = 0$, $X_1 = 0$, X_2 Free

$E_{-5} = \{(0, X_2, 0) \mid X_2 \in \mathbb{R}\}$

Span$\{(0,1,0)\}$

-5 appears once $\therefore$ 1 point.

$\alpha = 4$

$E_4 = Z(4I_3 - A) = Z \begin{bmatrix} 2 & 0 & -3 \\ 0 & 9 & -1 \\ 0 & 0 & 0 \end{bmatrix}$ $2X_1 = 3X_3 \rightarrow X_1 = 3/2\, X_3$, $9X_2 = X_3 \rightarrow X_2 = X_3/9$, X_3 free. OR JUST WORK ROW BY ROW

$E_4 = \{(\frac{3}{2}X_3, \frac{X_3}{9}, X_3)\} =$

Span$\{(\frac{3}{2}, \frac{1}{9}, 1)\}$ $\rightarrow$ I can write span$\{(27, 2, 18)\}$

4 appears once $\therefore$ 1 point

$\therefore$ Yes, diagnolizable

$$\begin{array}{ccc} E_{-5} & E_2 & E_4 \end{array} \quad \begin{array}{ccc} \alpha=-5 & \alpha=2 & \alpha=4 \end{array}$$

$$\underbrace{\begin{bmatrix} 0 & 1 & 3/2 \\ 1 & 0 & 1/9 \\ 0 & 0 & 1 \end{bmatrix}}_{Q} \underbrace{\begin{bmatrix} -5 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 4 \end{bmatrix}}_{D} Q^{-1} = A$$

Can be written $\begin{smallmatrix} 2 \\ 0 \\ 0 \end{smallmatrix} \rightarrow$ Span

D → can be written in any order

To verify find Q^{-1} then multiply 3 matrices and get A

Q. A 5x5, $C_A(\alpha) = (2+\alpha)^3(\alpha-5)^2$

$E_{-2} = \text{Span}\{(1,0,0,0,1), (0,1,0,0,0)\}$

$E_5 = \text{Span}\{(0,0,1,0,0), (0,0,0,0,1)\}$

Is A diagnolizable ?

eigenvalues of A $C_A(\alpha) = 0 \to \alpha = -2$ repeated 3 times

$\alpha = 5$ repeated 2 times

$\alpha = -2$ repeated 3 times, but $E_{-2} = \text{Span}\{2 \text{ points}\}$

A is not digonizable. We can't find 5x5 invertible matrix Q and diagonal matrix D {5x5} such that $QDQ^{-1} = A$

Know: A, nxn $\deg(C_A(\alpha)) = n$. A has at most n-eigenvalues

↳ degree of charectaristic polynomial

Ex. 3x3 never get 5 eigenvalues maximum 3

Q. A, 3x3 $C_A(\alpha) = (\alpha+5)^2(\alpha-10)$

$E_{-5} = \text{Span}\{(1,1,1),(0,0,1)\}$

$E_{10} = \text{Span}\{(0,2,4)\}$

① Is A diagonizable ? If yes, find invertible Q, diagonal D s.t. $QDQ^{-1} = A$

② Find A^{102}

① Eigenvalues $\alpha = -5$ (twice) $\alpha = 10$ (once)

$E_{-5} = \text{Span}\{2 \text{ points}\}$ $E_{10} = \text{Span}\{\text{one point}\}$ by class result, A is diagnolizable.

yes, E_{-5} E_{10} E_{-5} $\alpha=-5$ $\alpha=10$ $\alpha=-5$ ⇒ Can be in any order with repitition

$$\underbrace{\begin{bmatrix} 1 & 0 & 0 \\ 1 & 2 & 0 \\ 1 & 4 & 1 \end{bmatrix}}_{Q} \underbrace{\begin{bmatrix} -5 & 0 & 0 \\ 0 & 10 & 0 \\ 0 & 0 & -5 \end{bmatrix}}_{D} Q^{-1} = A$$

→ If I want the value of A, then, find Q^{-1} and multiply 3 matrices

I can put $\begin{smallmatrix}0\\20\\40\end{smallmatrix}$ any linear combination.

→ I can't choose the same point $\begin{smallmatrix}0\\0\\1\end{smallmatrix}$, two identical means inverse DNE } This why repeated 3 times I can't use the same point twice.

Know: A, nxn, if two rows or two columns are identical, then $|A| = 0$ hence A is singular, non invertible

Extra: $B = \begin{bmatrix} 1 & 0 & 1 \\ 10 & 3 & 10 \\ 20 & -5 & 20 \end{bmatrix}$ I claim $|B| = 0$ $|B| = |B^T|$

$$B^T = \begin{bmatrix} 1 & 10 & 20 \\ 0 & 3 & -5 \\ 1 & 10 & 20 \end{bmatrix} \xrightarrow{-R_1+R_3 \to R_3} \begin{bmatrix} & & \\ & & \\ 0 & 0 & 0 \end{bmatrix}$$

② If A is diagnolizable ∴ I can solve in 2 steps

$(QDQ^{-1})^2 = A^2$

$QDQ^{-1} \cdot QDQ^{-1} = A^2$

$QD^2Q^{-1} = A^2$

$A^3 = A^2A$

$= QD^2Q^{-1} \cdot QDQ^{-1} = QD^3Q^{-1}$

$A^4 = \quad = QD^4Q^{-1}$

$A^K = QD^KQ^{-1}$

↳ diagonal, how to multiply two diagonal matrices? Just multiply the #'s on the diagonal

$A^{102} = QD^{102}Q^{-1}$ ⇝ play with D, Q and Q^{-1} are the same

$$= \begin{bmatrix} 1 & 0 & 0 \\ 1 & 2 & 0 \\ 1 & 4 & 1 \end{bmatrix} \begin{bmatrix} (-5)^{102} & 0 & 0 \\ 0 & (10)^{102} & 0 \\ 0 & 0 & (-5)^{102} \end{bmatrix} Q^{-1}$$

Know: A, $n \times n$.

$|A|$ = multiplication of all eigenvalues of A with repetition

↳ If you know the eigenvalue, then you know the det

Ex. A, 3×3 $C_A(\alpha) = (\alpha+5)^2(\alpha-10)$

Find $|A|$? $(-5)^2(10) = 250$

Know:

① A is singular (non invertible) iff one of the eigenvalues is zero

② A is nonzero (invertible) iff zero is not an eigenvalue.

Def: Trace(A) = sum of all #s on the main diagonal

A, $n\times n$. Trace(A) = Sum of all eigenvalues of A (with repetition)

Ex. $A = \begin{bmatrix} 4 & 2 & 0 & 5 \\ 10 & 3 & 2 & 10 \\ 0 & 5 & 13 & 0 \\ 15 & 12 & 10 & 11 \end{bmatrix}$ Find Trace(A) = 4 * 3 + 13 + 11 = 31

If ~~10~~, 11, 20 are eigenvalues, what is the remaining?

↳ $-10 + 11 + 20 + X = 31$ (He'll give 3 and ask for the last eigenvalue)

Ex. A, 5×5 $C_A(\alpha) = (\alpha+3)^2(\alpha-7)^2(\alpha+11)$

1) Find Trace(A) 2) Find $|A|$ 3) Is A invertible

1) Trace(A) = $-3-3+7+7-11 =$

2) $|A| = (-3)^2(7)^2(-11) =$

3) yes, det is not zero

Q: A 4x4, $C_A(\alpha) = (\alpha+4)^3(\alpha-10)$

① Find the trace of A

$-4-4-4+10 = -2$

② Is it possible that $A = \begin{bmatrix} 3 & 0 & 2 & 1 \\ 4 & 10 & 1 & 1 \\ 0 & 10 & 2 & 1 \\ 12 & 10 & 8 & -18 \end{bmatrix}$

No, check the trace

$3+10+2-18 = -3$

$-3 \neq -2$

③ Find the det of A

$|A| = -4 \times -4 \times -4 \times 10 = -640$

④ Find $C_{A^T}(\alpha)$

Math know

$C_{A^T}(\alpha) = C_A(\alpha)$ they have the same eigen values

$CAT(\alpha) = C_A(\alpha)$

$\hookrightarrow C_{A^T}(\alpha) = |\alpha I_4 - A^T| = |(\alpha I_4 - A)^T| = |\alpha I_4 - A| = C_A(\alpha)$

⑤ Find the eigen values of A^3?

$(-4)^3$, $(10)^3$

" " " " " A^5?

$(-4)^5$, $(10)^5$

$A\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = 10\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix}$, $A^2\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = 10^2\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = 10 \times \left[A\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix}\right]$

⑥ If A^{-1} exist, find eigenvalues of A^{-1}

$\hookrightarrow$ det $\neq 0$ = -640

$\frac{-1}{4}$ (repeated 3 times), $\frac{1}{10}$ (repeated once)

$A \to \alpha = 2 \therefore A^{-1} \to \alpha = 1/2$

$A\begin{bmatrix} x_1 \\ \vdots \\ x_4 \end{bmatrix} = -4\begin{bmatrix} x_1 \\ \vdots \\ x_4 \end{bmatrix} \rightsquigarrow \begin{bmatrix} x_1 \\ \vdots \\ x_4 \end{bmatrix} = A^{-1}(-4)\begin{bmatrix} x_1 \\ \vdots \\ x_4 \end{bmatrix} \rightsquigarrow \frac{-1}{4}\begin{bmatrix} x_1 \\ \vdots \\ x_4 \end{bmatrix} = A^{-1}\begin{bmatrix} x_1 \\ \vdots \\ x_4 \end{bmatrix}$

⑦ Find the eigen value of $7A$

$7(-4) \to$ repeated 3 times $7(10) \to$ repeated once

⑧ Find $|A - 2I_4| \rightsquigarrow$ Find eigen values then the det. I know d, I can get α

eigen values of $A - 2I_4 \to \underset{(3\ \text{times})}{-4-2}$, $\underset{(\text{once})}{10-2}$ $\quad -6^3 \times 8 \to$ det

" $|A + 3I_4|$

$\underset{(3\ \text{times})}{-4+3}$, $\underset{(\text{once})}{10+3}$

Wrong! common mistake

$$A = \begin{bmatrix} 0 & 3 & 2 \\ 1 & 4 & 0 \\ 0 & 0 & 10 \end{bmatrix} \xleftrightarrow{R_1 \leftrightarrow R_2} \begin{bmatrix} 1 & 4 & 0 \\ 0 & 3 & 2 \\ 0 & 0 & 10 \end{bmatrix} = B$$

eigenvalues of A $\neq$ eigen values of B

If A and B are equiuilant eigenvalues of A need not same as eigen values of B

Ex. $A = \begin{bmatrix} 1 & 0 \\ 0 & 4 \end{bmatrix} \xrightarrow{\frac{-1}{3}R_2} B = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

eigen values of A

$C_A(\alpha) = (\alpha-1)(\alpha-4) \therefore \alpha = 1 \quad \alpha = 4$

eigen values of B

$C_B(\alpha) = (\alpha-1)(\alpha-1) \therefore \alpha = 1$ twice or directly diagonal Mat.

Result: No row operations! either defn ~~or row operations~~

$A\begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix} = \alpha \begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix}$ $(a_1, a_2, a_3) \neq (0,0,0)$ defn to be eigen, at least 1 Point $\neq$ all zero's

$$(A-2I_3)\begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix} = A\begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix} - \underbrace{2I_3}_{\text{identity}}\begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix} = \alpha\begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix} - 2\begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix} = (\alpha-2)\begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix}$$

A 4×4 $C_A(\alpha) = (\alpha-1)(\alpha+2)(\alpha-4)^2$ $\alpha = 1, -2, 4$ (twice)

$|A+3I_4|$ what are α's? $1+3$, $-2+3$, $4+3$ (twice)

$|A+3I_4| = 4 \times 1 \times 7 \times 7 =$

trace $= 4 + 1 + 7 + 7$

Q: What does diagnolizable mean?

we have $QDQ^{-1} = A$ where D is a diagonal matrix, Q^{-1} invertible matrix

Know: assume A $n\times n$ without this word, wrong

assume A has n $\boxed{\text{distinct (different)}}$ eigenvalues. Then A is diagnolizable

In fact if α is an eigenvalue of A then $E_\alpha = \text{Span}\{\text{one point}\}$

No rept.

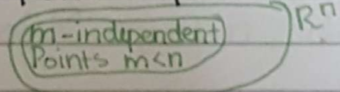
Q: $m = \text{Span}\{(1,4,2),(0,1,7),(1,8,30)\}$
Find independent # of M (dem(M))
↳ As if asking: (↳ from dimention)
How many points in the span are independent?

$$\begin{bmatrix} 1 & 4 & 2 \\ 0 & 1 & 7 \\ 1 & 8 & 30 \end{bmatrix} \xrightarrow{-R_1+R_3} \begin{bmatrix} 1 & 4 & 2 \\ 0 & 1 & 7 \\ 0 & 4 & 28 \end{bmatrix} \xrightarrow{-4R_2+R_3} \begin{bmatrix} 1 & 4 & 2 \\ 0 & 1 & 7 \\ 0 & 0 & 0 \end{bmatrix}$$ kill below only

(1,4,2) (0,1,7) (1,8,30) are dependent, and since I have 2 leader's ∴ two of the points are independent (1,4,2) and (0,1,7). The 3rd point depends on the other two points.
If one point depends on the other two this means it can be written as a linear combination ∴ I can remove it from the span since I have ∞ # of points ∴ Span = {(1,4,2) (0,1,7)}
The independent # / dimention is 2

Know: in R^n maximum # of independent points is n
(Without calculations) 4 points in R^3
3 leaders and one is written as a linear combination ∴ dependent
7 points in R^6
6 leaders ∴ dependent
Max of independent points in R^4? 4 ∴ column # = ind

Big result
① R^n = Span {n-independent points}
② If m<n, then $R^n \neq$ Span {m-independent points}
Span {m-independent points in R^n} "lives" inside R^n but not equal to R^n



③ Any n independent points in R^n will span R^n
Choose any n independent points in R^n, say $Q_1, Q_2 \cdots Q_n$ then span $\{Q_1, Q_2 \cdots Q_n\} = R^n$

Ex. $\mathbb{R}^4$

$(0,0,1,0,0)$ $(0,0,0,0,1)$

$\mathbb{R}^4 = \text{span}\{(1,0,0,0,1),(0,1,0,0,0),(0,0,1,0),(0,0,0,1)\}$

$Q = L = \text{Span}\{(1,2,0,1),(-1,-1,1,-1),(-1,-2,1,3)\}$

a. Is $L = \mathbb{R}^4$?

No, by class result

b. Find independent point of L ($\dim(L)$)

$$\begin{bmatrix} 1 & 2 & 0 & 1 \\ -1 & -1 & 1 & -1 \\ -1 & -2 & 1 & 3 \end{bmatrix} \xrightarrow[R_1+R_3 \to R_3]{R_1+R_2 \to R_2} \begin{bmatrix} 1 & 2 & 0 & 1 \\ 0 & 1 & 1 & 2 \\ 0 & 0 & 1 & 4 \end{bmatrix}$$ all 3 points are independent

$L = \text{Span}\{(1,2,0,1),(0,1,1,2),(0,0,1,4)\}$

So, independent no. $= 3$ ($\dim(L) = 3$)

c. Find a basis for L

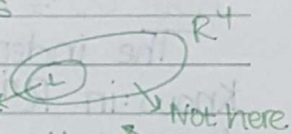
The independent points

basis for $L = \{(1,2,0,1), (-1,-1,+1,-1)(-1,-2,1,3)\}$

any 3 independent points in L $\to$ will form a basis

$L = \text{Span}\{$any 3 independent points$\}$

$L = \text{Span}\{(1,2,0,1), (0,1,1,2), (0,0,1,4)\}$ Points live here



$\hookrightarrow$ By calculations you get new points, you can use them

Big result:

$L = \text{Span}\{Q_1, Q_2, Q_3 .. Q_n\}$ assume only Q_1, Q_3, Q_5 are independent.

Any 3 independent points in L will form a basis

$L = \text{Span}\{$any independent points$\}$

Vector space

$(V, +, \cdot)$ is called vector space if

set; scale multiplication

① $0 \to (0,0 .. 0)$ if live in $\mathbb{R}^n$

$\to$ ~~0 belongs to V~~

$\to 0 + 0x + 0x^2 + .. 0x^m$

$\to \begin{bmatrix} 0 & 0 & .. & 0 \\ 0 & 0 & .. & 0 \\ 0 & 0 & .. & 0 \end{bmatrix}$ if live inside matrices

(Zero matrix)

② Closure under addition set

Choose any two elements in V, if we add them we should stay inside V

$(V_1, V_2 \in V \xrightarrow{\text{implies}} U_1 + U_2 \in V)$

belongs; belong

③ Closure under multiplication

Choose any element in V and multiply it by a scallar # then we should stay insider ($V_i \in V, \alpha \in R \rightarrow \alpha V_i \in V$)

To prove if is a vector space → these ③ points should be satisfied.

Show $D = \text{span}\{(1,4,0), (-1,1,2)\}$ is a vector space.

① $O = (0,0,0) \in D$

yes, lives in R^3, $0(1,4,0) + 0(-1,1,2) = (0,0,0)$ any linear combination lives in D

② $V_1 = \alpha_1(1,4,0) + \alpha_2(-1,1,2)$ for some $\alpha_1, \alpha_2 \in R$

$V_2 = B_1(1,4,0) + B_2(-1,1,2)$ for some $B_1, B_2 \in R$

$V_1 + V_2 = (\alpha_1 + B_1)(1,4,0) + (\alpha_2 + B_2)(-1,1,2) \in D$

Selected two things in D and added them in D

③ Choose $a \in R$, $V \in D$

$aV_1 = a\alpha_1(1,4,0) + a\alpha_2(-1,1,2) \in D$

Big result:

① Span{of points} is a vector space

② vector space 1,2,3 are satisfied

The soln set to the homogeneous system is a vector space

$E_3 = Z(3I_n - A)$ → zeros ∴ homogeneous ∴ 3 axium sat.

E_3 is a vector space

↳ Two points in E_3 add them I'll stay in E_3

↳ Two points that live in the homogen will give a point that lives in same D

Soln set for nonhomogeneous will never be a vector space.

Result: Observe

① Independent # of R^n equal n, $\dim(R^n) = n$

If R^5

$D = \text{Span}\{(1,1,1,1,0)\ (0,-1,2,3,0)\}$ lives in R^5 but not equal, to be equal we should have 5 independent points

2 points → plane

1 point → line

for 3×5 soln set lives in ? R^5

We look at columns not rows

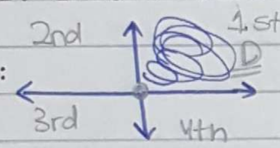
Big result (main result on vector spaces) :

$(V,+,\cdot)$ → Live in R^n point
→ Live in matrices $R^{n\times m}$
→ Live in polynomials P^n

V is a vector space iff V = Span{points or matrices or polynomials}

Know R^n is a vector space
If I don't want to use this result use 1, 2, 3 (3 axium)

Q:



D = Set of all points in the 1st quad. of $\mathbb{R}^2$
Claim not a vector space ∴ one of 3 axiums is not satisfied.
① $0 = (0,0) \in D$ → From the plane
② $V_1, V_2 \in D \rightarrow V_1 + V_2 \in D$ Subset of R^2 ↓ lives in it
$+x$ $+y$ (From +ve part) → Two points live in D Subspace of R^2 ✗ Not vector spa.
③ $(1,4) \in D$ $\alpha = -1$
$-(1,4) = (-1,-4) \notin D$ Fail
∴ Not a vectorspace

Def :
A vector space D is called a subspace of F if F is a vector space and D lives inside F

D vector space → F vectorspace, D subspace of F

we have to test subspace → always subset but

Q: Is $L = \{(x_3 - x_4, -3x_4, x_3, x_4) \mid x_3, x_4 \in R\}$ a subspace of $\mathbb{R}^4$?
↳ we can have a system of nonlinear eqns not only a span

Check if I can write it ~~in a span~~ as a span
$L = \{x_3(1,0,1,0) + x_4(-1,-3,0,1) \mid x_3, x_4 \in R\}$
$L = \text{Span}\{(1,0,1,0), (-1,-3,0,1)\}$
Check ①②③ they are satisfied. yes, vector space
independent $\# L = \dim(L) = 2$ independent points = Free variables
For homo. in linear eqn
↳ Can be written as a span of Points like # free variable
Only true for
add to the soln set of homo geneous system

Know:
Solution set of a homogeneous system equal
Span{K points where K is # of free variables}
independent (always without checking)

Example: $L = \{(a+2b, 3a+6b, 0) \mid a,b \in R\}$
$L = \{a(1,3,0) + b(2,6,0) \mid a,b \in R\}$
$L = Span\{(1,3,0), (2,6,0)\}$
Set live in R^3, independent # is 1

$$\begin{bmatrix} 1 & 3 & 0 \\ 2 & 6 & 0 \end{bmatrix} \xrightarrow{-2R_1+R_2 \to R_2} \begin{bmatrix} 1 & 3 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

Base and independent #
independent points — tell how many are ind.
Basis for $L = \{(1,3,0)\}$ L=
Both dependent, choose any one as a base

P_n = Set of all polynomials of degree $<n$
Polynomial: $a_n x^{n-1} + a_{n-1} x^{n-1} + \dots + a_1 x + a_0$
$a_0, a_1, \dots a_n \in R$ — whole #
but all exponents must be +ve integers

Is the following a polynomial:
$\sqrt{x} + 3x - 2$ X $x^{1/2} + 2x$ X $\frac{2}{x^2+1}$ X X $-3x^{-2} + x$ X $10x^4 - x^2 + 13$ ✓
$10x^4 - x^2 + 13 \to$ degree 4
$\hookrightarrow$ Does it $\in P_4$ NO

$f(x) = 0 \to$ degree 0 $f(x) = 2 \to$ degree 0 $f(x) = 2x \to$ degree 1
P_3 means: Set of all polynomials that are of degree <3
$P_3 = \{a_2x^2 + a_1x^1 + a_0 \mid a_0, a_1, a_2 \in R\}$ — generally
$P_n = \{a_{n-1}x^{n-1} + a_{n-2}x^{n-2} + \dots + a_1x + a_0 \mid a_0 \dots, a_{n-1} \in R\}$

Check if P_n is a vector space
A) * P_n, $O = 0 \to$ normal zero (all a are zero's)
P_{3}, $O = 0x^2 + 0x + 0 = 0 \to$ degree zero $0<n$
* closure under addition (same degree)
* " " multiplication (multiply by const, same degree)

B) Or check span
$P_3 = Span\{x^2, x, 1\}$ ∴ span of 3 things, linear combination give poly
ex. $a_2x^2 + a_1x + a_0$
Independent can't be written as linear combination

$R^5 \to$ can it be a span of 4 points?

independent # of P_3, $\dim(P_3) = 3$
If I have 4 Polynomials $\therefore$ dependent
Span of 2 polynomials live in P_3 but not equal to P_3
I need 3 ind. polynomials to be equal
P_n is a vector space
$P_n = \text{Span}\{x^{n-1}, x^{n-2}, \dots x, 1\}$ I have n polynomials (elements)
they have different degrees so can't be a linear combination $\therefore$ independent
independent # of $P_n = 0$
P_7: ① Independent # $= 7$
② $P_7 = \text{Span}\{$any 7 ind. polynomials in $P_7\}$
③ any 8 polynomials are dependent
Set of polynomials of degrees 6,5,4,3,2,1,0
basis consist of 7 independent polynomials
We have $R^n \to$ points $P^n \to$ polynomials $R^{n\times m} \to$ matrices
$R^{n\times m}$ = Set of all $n\times m$ matrices
$R^{3\times 2}$ = Set of all 3×2 matrices
$= \left\{ \begin{bmatrix} a_1 & a_2 \\ a_3 & a_4 \\ a_5 & a_6 \end{bmatrix} \middle| \; a_1, a_2, a_3, \dots a_6 \in \mathbb{R} \right\}$
Ⓐ
Span $\leftarrow$ 3axiums
Is it a vector space? $\left\{ a_1 \begin{bmatrix} 1 & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix} + a_2 \begin{bmatrix} 0 & 1 \\ 0 & 0 \\ 0 & 0 \end{bmatrix} + \dots + a_6 \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 1 \end{bmatrix} \middle| \; a_1, a_2 \dots a_6 \in \mathbb{R} \right\}$

$\text{Span}\left\{ \begin{bmatrix} 1 & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 1 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 1 \end{bmatrix} \right\}$

any 3x2 matrix is a linear combination of these 6.
$=$ Span$\{$ of 6 independent 3x2 matrices $\}$
$\dim(R^{3\times 2}) = 6$
Ⓑ $0 = \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}$, P^n $0 = 0 \#$, R^n, $0 = (0, 0, \dots 0_n)$ $\rightsquigarrow$ depends on where I live

$R^{5\times 10}$ = Span$\{$50 independent 5×10 matrices$\}$ $\to$ 51 $\therefore$ dependent
$\to$ Can't be written as a span of less than 50
$R^{n\times m}$ = Span$\{nm$ independent $n\times m$ matrices$\}$
is a vector space.

a huge result: P_n as vector space is the same as R^n ①
is isomorphic to as a vector space

$R^{n\times m}$ as vector space is the same as R^{nm} ②

*"P_n is the same as R^n" WRONG! should write vector space.

① P_n as vector space is the same as R^n

$a_{n-1}x^{n-1} + a_{n-2}x^{n-2} + \dots a_1x + a_0 \longleftrightarrow (a_{n-1}, a_{n-2}, \dots a_1, a_0)$

P_4 as vector space is the same as R^4

$0x^3 + 0x^2 + 4x + 3 \longleftrightarrow (0,0,4,3)$ # Write poly in descending order, highest to lowest power

$x^3 + 3x^2 + 2 \longleftrightarrow (1,3,0,2)$

② $R^{n\times m}$ as vector space is the same as R^{nm}

$R^{3\times 2} \longleftrightarrow R^6$

$\begin{bmatrix} a_1 & a_2 \\ a_3 & a_4 \\ a_5 & a_6 \end{bmatrix} \quad (a_1, a_2, a_3, a_4, a_5, a_6)$

$\begin{bmatrix} 3 & 0 \\ 2 & 1 \\ 0 & 5 \end{bmatrix} \longleftrightarrow (3,0,2,1,0,5)$

$\begin{bmatrix} 0 & 2 \\ 1 & 0 \\ 4 & 1 \end{bmatrix} \longleftrightarrow (0,2,1,0,4,1)$

$D = \text{Span}\{x^2+1, -2x, 2x^2-4x+2, x^2-4x+1\}$

① is D a subspace of P_3? → Check If the Q span if ✓ then subspace
② Find independent # of D, $\dim(D)$. If no write it as a span if
③ Find a basis for D You can then subspace.
④ $2x^2-6x+2$ belong to D? Can you write it as a linear combination?

① yes, because $D = \text{Span}\{\quad\}$
If you can write it a span ∴

② P_3 → max span of 3 #s to be dependent, but here 4 ∴ dependent.
(Can be 1, 2, or 3)

$P_3 \longleftrightarrow R^3$

P_3	R^3
x^2+0x+1	$(1,0,1)$
$0x^2-2x+0$	$(0,-2,0)$
$2x^2-4x+2$	$(2,-4,2)$
x^2-4x+1	$(1,-4,1)$

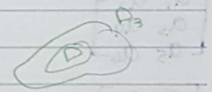
$$\begin{bmatrix}1&0&1\\0&-2&0\\2&-4&2\\1&-4&1\end{bmatrix} \xrightarrow[-R_1+R_4\to R_4]{-2R_1+R_3\to R_3} \begin{bmatrix}1&0&1\\0&-2&0\\0&-4&0\\0&-4&0\end{bmatrix} \xrightarrow{-\frac{1}{2}R_2} \begin{bmatrix}1&0&1\\0&1&0\\0&-4&0\\0&-4&0\end{bmatrix}$$

$$\xrightarrow[4R_2+R_4\to R_4]{4R_2+R_3\to R_3} \begin{bmatrix}1&0&1\\0&1&0\\0&0&0\\0&0&0\end{bmatrix}$$

∴ independent # of $D = 2$
I didn't interchange any row ~~so same polynomial~~

③ Since the input is polynomial ∴ output is polynomial $= \{(1,0,1)(0,-2,0)\}$ WRONG
another basis go to row operation and choose any 1st two

Basis for $D = \{x^2+1, -2x\}$ or $= \{x^2+1, x\}$
D lives inside P^3, $P^3 \neq D$, they have different dim



④ always solve this Q at the end
$D = \text{span}\{\quad\} = \text{span}\{x^2+1, x\}$ ← easier to solve 4 no extra terms

Check if $(2,-6,2)$ belong $\in$ span$\{x^2+1, x\}$ | span$\{(1,0,1)(0,1,0)\}$
$(2,-6,2) = \alpha_1(1,0,1) + \alpha_2(0,1,0)$

Find $\alpha_1 = 2$ $\alpha_2 = -6$ → Check the last # → $2 = \alpha_1 + 0\alpha_2 \to \alpha_1 = 2$
$\alpha_2 = -6$ last point
$2(1) - 6(0) = 2$ ✓

I can find α_1, α_2 ∴ yes it $\in D$
α_1, α_2 for point is the same as for poly

If the eqn in ④ was $(2x^2-6x-2)$
$\alpha_1 = 2$ $\alpha_2 = -6$ → Check last # $(2)(1) - 6(0) = 2 \neq -2$ ∴ NO.

Q $D = \{f(x) \in P_4 \mid f'(0) = 1\}$ is D a subspace (vector space) of P_4

$f(x) = x^3 + x \in P_4$ $f'(0) = 1 \in D$

① poly degree is less than 4 ② 1st derivative is 1 (at zero) $f'(x) = 3x^2 + 1$

∴ yes, lives inside.

$f(x) = x^3 + 2x^2 - 1$

① Poly degree is less than ④ ② $f'(x) = 3x^2 + 4x$ at $f'(0) = 0 \neq 1$ NO $\notin D$

∴ No, it doesn't live inside.

All axioms are wrong in this case (just show one)

$f_1(x) = x \in D$, derivative $= 1$

$f_2(x) = x^2 + x \in D$, derivative $= 1$

Add $x^2 + 2x$, derivative $= 2 \notin D$ 2nd axium fails

Multiply $f(x) = x$, $-1 \cdot f(x) = -x$, derivative $= -1 \notin D$ 3rd axium fails

Q $D = \left\{ \begin{bmatrix} a+b & -2a & b \\ c & -2a-2b & -4c \end{bmatrix} \middle| a,b,c \in \mathbb{R} \right\}$

① is D a subspace of $\mathbb{R}^{2\times3}$?

② Find independent # of D, dim(D)?

③ Find a basis for D.

① always start with the easiest 0.

0 in $\mathbb{R}^{2\times3} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \in D$ yes $a = b = c = 0 \to$ can't check addition/multiplication or just check span

if $D = \left\{ \begin{bmatrix} a+b & -2a & b+1 \\ c & -2a-2b & -4c \end{bmatrix} \middle| a,b,c \in \mathbb{R} \right\}$

0 is not satisfied $\begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$ ∴ NO

Find span

$\left\{ a\begin{bmatrix} 1 & -2 & 0 \\ 0 & -2 & 0 \end{bmatrix} + b\begin{bmatrix} 1 & 0 & 1 \\ 0 & -2 & 0 \end{bmatrix} + c\begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & -4 \end{bmatrix} \middle| a,b,c \in \mathbb{R} \right\}$

$= \text{Span}\left\{ \begin{bmatrix} 1 & -2 & 0 \\ 0 & -2 & 0 \end{bmatrix}, \begin{bmatrix} 1 & 0 & 1 \\ 0 & -2 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & -4 \end{bmatrix} \right\} \to$ max is 3

Yes, why

② $\mathbb{R}^{2\times3} \to$ max is 6, but we check for dim in $\mathbb{R}^6$

$\begin{bmatrix} 1 & -2 & 0 & 0 & -2 & 0 \\ 1 & 0 & 1 & 0 & -2 & 0 \\ 0 & 0 & 0 & 1 & 0 & -4 \end{bmatrix} \xrightarrow{-R_1+R_2 \to R_2} \begin{bmatrix} 1 & -2 & 0 & 0 & -2 & 0 \\ 0 & 2 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 & -4 \end{bmatrix}$ 3 leaders

dim(D) = 3

③ basis $\left\{ \begin{bmatrix} 1 & -2 & 0 \\ 0 & -2 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 2 & 1 \\ 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 0 \\ -1 & 0 & -4 \end{bmatrix} \right\}$ add row operation $\frac{1}{2}R_2$

a different basis $= \left\{ \begin{bmatrix} 1 & -2 & 0 \\ 0 & -2 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 & 1/2 \\ 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & -4 \end{bmatrix} \right\}$

Show $D = \{ f(x) \in P_4 \mid \int_0^1 f(x) = 0$ is a subspace of $P_4 \}$
Find $\dim(D)$ and a basis for D

Read the 2 Q's it should be written as a span ∴ rewrite the Q

$$D = \{ a_3x^3 + a_2x^2 + a_1x^1 + a_0 \mid \frac{a_3x^4}{4} + \frac{a_2x^3}{3} + \frac{a_1x^2}{2} + a_0x \Big|_0^1 = 0 \} \therefore$$

$$= \{ a_3x^3 + a_2x^2 + a_1x^1 + a_0 \mid \frac{a_3}{4} + \frac{a_2}{3} + \frac{a_1}{2} + a_0 = 0 \}$$

One eqn can be written where 1 is leader ∴ The rest are free.

$$= \{ a_3x^3 + a_2x^2 + a_1x^1 + a_0 \mid \frac{a_0}{1} = -\frac{a_3}{4} - \frac{a_2}{3} - \frac{a_1}{2}, a_1, a_2, a_3 \in \mathbb{R} \}$$

$$= \{ a_3x^3 + a_2x^2 + a_1x - \frac{a^3}{4} - \frac{a_2}{3} - \frac{a_1}{2} \mid a_1, a_2, a_3 \in \mathbb{R} \}$$

$$= \text{Span} \{ a_3(x^3 - \tfrac{1}{4}) + a_2(x^2 - \tfrac{1}{3}) + a_1(x - \tfrac{1}{2}) \mid a_1, a_2, a_3 \in \mathbb{R} \}$$

$$= \text{Span} \{ x^3 - \tfrac{1}{4}, x^2 - \tfrac{1}{3}, x - \tfrac{1}{2} \}$$ ← 3 poly's of different degrees ∴ independent
mutiply by scaller won't change the power as well addition won't change.

* Any polynomial of degree < 4 and $\int_0^1 f(x) = 0$ can be written in this form
* To check integrate and sub directly for every 1 of 3.

$\text{Dim}(D) = 3$

basis of $D = \{ x^3 - \frac{1}{4}, x^2 - \frac{1}{3}, x - \frac{1}{2} \}$

Q: $(1,2,3) \in \mathbb{R}^3$. $D = \{ (a_1, a_2, a_3) \in \mathbb{R}^3 \mid a_1 + 2a_2 + a_3 = 0 \}$

D is a subspace ∴ write it as a span, come with leader the rest are free

$D = \{ (-2a_2 - a_3, a_2, a_3) \mid a_2, a_3 \in \mathbb{R} \} = \{ a_2(-2,1,0) + a_3(-1,0,1) \}$ ∴

Span $\{ (-2,1,0)(-1,0,1) \}$ YES! IT IS A SUBSPACE.

$\begin{bmatrix} -2 & 1 & 0 \\ -1 & 0 & 1 \end{bmatrix}$ both will survive ∴
$\dim(D) = 2$ basis:

H.W $D=\{f(x)\in P_4 \mid f'(1)=0\}$

① Show D is a subspace of P_4

② Find independent #

③ Find a basis

$\{a_3X^3+a_2X^2+a_1X+a_0 \mid 3a_3X^2+2a_2X+a_1\}$

$\{ \qquad \mid 3a_3+2a_2+a_1=0\}$

$\{ \qquad \mid a_1=-3a_3-2a_2, a_3,a_2,a_1\in\mathbb{R}\}$

$\{a_3X^3+a_2X^2+(a_1X-3a_3-2a_2)X + a_0 \mid a_3,a_2\in\mathbb{R}\}$

$\{a_3(X^3-3)+a_2(X^2-2)+a_1(X) \mid \}$

Span $\{X^3-3, X^2-2, X\}$

$\cancel{X^3 \; -3}$

$\cancel{X^2 \; -2}$

X

HW $D=\{(a_1,a_2,a_3,a_4)\in\mathbb{R}^4 \mid a_1+a_4=0,\ a_2+a_3-a_4=0\}$ → Homo. ∴ ind # = Free variables #

Show D is a subspace of $\mathbb{R}^4$, Find Dim(D), find basis

Find soln set to homogeneous system → soln set consist of points in $\mathbb{R}^4$ where the eqn is satisfied.

$a_1=-a_4 \qquad a_2=a_4+a_3$

$=\{(-a_4, a_4+a_3, a_3, a_4) \mid a_3,a_4\in\mathbb{R}\}$

$=\{a_4(-1,1,0,1)+a_3(0,1,1,0) \mid \}$

Span $\{(-1,1,0,1),(0,1,1,0)\}$

Q. $A = \begin{bmatrix} 1 & -1 & 0 & 3 & 4 \\ -1 & 1 & 0 & -2 & 0 \\ 2 & -2 & 0 & 6 & 8 \end{bmatrix}$ $R^{3\times5}$

1. Find Rank(A) → Since I wrote basis ∴ vector space
2. Find a basis for Row(A)
3. Find a basis for column(A)

Row(A) = Span$\{R_1, R_2, R_3\}$ = Span{(1,-1,0,3,4) (-1,1,0,-2,0) (2,-2,0,6,8)} R^3

Col(A) = Span$\{C_1, C_2, C_3, C_4, C_5\}$ = Span{(1,-1,2) (-1,1,-2) (0,0,0) (3,-2,6) (4,0,8)}

* Find ind. to find basis

Rank(A) = Independent # of row(A) dim(Row(A))

*Directly start working on A

$$A \xrightarrow[-2R_1+R_3 \to R_3]{R_1+R_2 \to R_2} \begin{bmatrix} 1 & -1 & 0 & 3 & 4 \\ 0 & 0 & 0 & 1 & 4 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

Rank(A) = 2
↳ Two rows survived

Basis for Row(A) = {(1,-1,0,3,4) (0,0,0,1,4)}

Know:

Independent # of Col(A) = Independent # of Row(A).

∵ I have 2 independent rows ∴ 2 columns. If 3 columns are independent ∴ I have 3 ~~columns~~ rows only

→ Not a linear combination of columns only linear combination of rows. But for sure 1st, 4th are linear combination of columns∴

Basis for col(A) = {(1,-1,2) (3,-2,6)}

Row(A) = Span{(1,-1,0,3,4) (0,0,0,1,4)} → When you know the basis of row space all the other rows are linear comb.

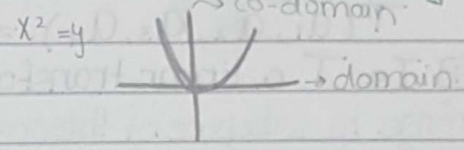
Col(A) = Span{(1,-1,2) (3,-2,6)} → Was 5, became 2. the other are linear comb.

HW $A = \begin{bmatrix} 0 & 0 & 1 & 2 & 3 \\ 1 & 1 & 0 & 4 & 5 \\ -1 & -1 & 0 & -4 & -4 \end{bmatrix}$ (a) Rank(A), (b) ~~linear~~ basis for Row(A)
(c) basis for Col(A)

a linear transformation.

Linear Transformation → Every linear transformation is a fn but not every fn

Q $T: \mathbb{R}^3 \longrightarrow \mathbb{R}^2$
domain co-domain → range
range



$T(a_1, a_2, a_3) = (-a_1, a_2 - a_1)$

he'll never ask this Q → Always linear trans.

ⓐ Is T a linear transformation

ⓑ Find a basis for range of T

ⓒ Find the zeros of T

range like in co-domain but neednot to be equal to it

ⓐ T is a linear transformation iff Range(T) is a subspace of the co-domain

Range(T) $= \{(-a_1, a_2 - a_1) \mid a_1, a_2 \in \mathbb{R}\}$ → Is it vector space? write it as a span

Range(T) $= \{a_1(-1,-1) + a_2(0,1) \mid a_1, a_2 \in \mathbb{R}\}$

$= \text{Span}\{(-1,-1)\ (0,1)\}$ yes, T is a linear transformation

* Look at the range and if I can write it as a subspace of the co-domain.

* Only Nul I don't check if they are linear combination, otherwise check if dependent or independent.

By simple calculations basis for Range(T) $= \{(-a_1, a_2 - a_1) \mid a_1, a_2 \in \mathbb{R}\}$

$= \{(-1,-1), (0,1)\}$

In this question, we can choose another basis (NOTE: In the Q Range(T) $= \mathbb{R}^2$ }

$\mathbb{R}^2 \rightarrow n = 2$ ∴ Any 2 independent points in $\mathbb{R}^2$. If $\mathbb{R}^4$ $n=4$ ∴ 4 independent points

ⓒ Another name: Ker(T), nul space of T) → points in $\mathbb{R}^3$

Set $T(a_1, a_2, a_3) = 0 = (0,0,0)$

Find the points (a_1, a_2, a_3) → Homogeneous

$(-a_1, a_2, -a_1) = (0,0) \rightarrow [-a_1 = 0,\ a_2 - a_1 = 0]$

$a_3 \in \mathbb{R}$ → has nothing to do, if it changed, it won't affect the answer }

Solve zeros for fn
co-domain
$y = 0$ → Take fn $= 0$
Point in co-domain
any answer x

?

Soln set to the homo. sys. as shown is zeros of T.

$\{(0, 0, a_3) \mid a_3 \in \mathbb{R}\}$ → Can be written as span

Span$\{(0,0,1)\}$

domain $\qquad$ co-domain

Q: $T: \mathbb{R}^4 \longrightarrow \mathbb{R}^3$ $\quad$ 1st R $\quad$ 2nd R $\quad$ 3rd R $\quad$ The image of the points

$T(a_1, a_2, a_3, a_4) = (a_1 - a_3, a_3, -a_1)$ $\quad$ is found by the formula

he'll never ask this Q. Always Linear trans.

[1] Is T a linear transformation

range is subspace of the co-domain $\text{Range}(T) = \{(a_1 - a_3, a_3, -a_1) \mid a_1, a_3 \in \mathbb{R}\}$

$\text{Range}(T) = \text{Span}\{(1,0,-1)(-1,1,0)\}$

So, T is a linear transformation

[2] Find the standard matrix representation of T

Linear transformation can be represented as a matrix and I can get all info. from that matrix. $\Rightarrow$ Whatever Q, the answer is in the matrix

(a_1, a_2, a_3, a_4) are points in $\mathbb{R}^4$ (the domain)

↳ Rely on 4 variables

$$\underset{3\times4}{M} = \begin{array}{cccc} a_1 & a_2 & a_3 & a_4 \\ \end{array} \begin{bmatrix} 1 & 0 & -1 & 0 \\ 0 & 0 & 1 & 0 \\ -1 & 0 & 0 & 0 \end{bmatrix}$$

Standard matrix representation

This M means:

$T(a_1, a_2, a_3, a_4) = \underset{3\times4}{M} \underset{4\times1}{\begin{bmatrix} a_1 \\ a_2 \\ a_3 \\ a_4 \end{bmatrix}} = \underset{3\times1}{\begin{bmatrix} \; \end{bmatrix}}$ view it in $\mathbb{R}^3$

Image of point =

Now no need for the formula

Know:

M independent # of codomain {multiplied} independent # of domain

3 eqns from codomain 3 $\qquad$ depends on unknown 4

Example of M

$T(2,1,3,4) = M \begin{bmatrix} 2 \\ 1 \\ 3 \\ 4 \end{bmatrix}$

If I have $\mathbb{R}^2 \to \mathbb{R}^5$

$\begin{bmatrix} a_1 & a_2 \\ & \\ & \end{bmatrix}$

[3] Find the zero's of T → Things in domain with image = 0 ↝ like $y=0$ x int = ?

[4] Find basis for Range T

zero's of T (another name Ker(T), nul space of T)

$T(a_1, a_2, a_3, a_4) = \underset{3\times4}{M} \underset{4\times3}{\begin{bmatrix} a_1 \\ a_2 \\ a_3 \\ a_4 \end{bmatrix}} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$ ↝ Soln set for homogeneous system.

Soln set for homogeneous system $Z(M) = N(M)$

Know: $Z(T) = Z(M)$ $\quad$ 1st C $\quad$ 3rd C

$$\begin{bmatrix} 1 & 0 & -1 & 0 \\ 0 & 0 & 1 & 0 \\ -1 & 0 & 0 & 0 \end{bmatrix} \xrightarrow{R_1 + R_3 \to R_3} \begin{bmatrix} 1 & 0 & -1 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & -1 & 0 \end{bmatrix}$$

$a_3 = 0$

$a_1 = a_3 = 0$

$a_2, a_4 \in \mathbb{R}$

zero's of $T = Z(M) = \{(0, a_2, 0, a_4) \mid a_4, a_2 \in \mathbb{R}\}$ → Since homo. i. can be written as a span

$\text{Span}\{(0,1,0,0)(0,0,0,1)\}$ → Soln set of homo. without checking always independent!

only points or any linear comb makes range = 0

$Z(T) = Z(M) \to$ Soln set for any homo. is span
↳ vector space or subspace

Know:
$Z(T), Z(M)$ always a subspace of the domain

Know:
$\underbrace{\text{Range}(T)} = \underbrace{\text{Col}(M)}_{\text{span of columns of } M.} = \text{Span}\{(1,0,-1)(-1,1,0)\}$ ← we found it before but here no need to check which is dep. or indp.

$M\begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix}$ ← linear comb. of columns of M

Basis for range $= \{(1,0,-1)(-1,1,0)\}$
Ind # range = 2
Ind # Z(T) = 2
→ 2 + 2 = 4 NOT accident = ind # of domain

* Rank = Ind # row = Ind # ~~domain~~ col
* Ind # of domain = Ind # Z(T) + Ind # Range(T) ← Big result
  ↳ If I know one, I should know the others.
* 4 var in M → 4 points in domain. Or, If I have 5 points in the domain I know I have 5 var.
* Ind # Z(T) = Ind # Z(M) → depends on how many free variables
If 4 variables → 2 Free and the rest (2) leaders

How many variables in M {in domain}

Free → dimention of zeros of (T) = # of free variables

Leading → dimention of range = # leading variables

ADD THEM = # of all variables ← depends on the domain.

In prev. question's related to {Show that T is a linear transformation}
↳ We have to assume that it is a linear transformation. He'll never ask to show.

* If you have a linear transformation then the range is a vector of subspace but if the range is subspace we don't know if it is a vector space.

Given $T: \mathbb{R}^4 \to \mathbb{R}^5$

$T(a_1,a_2,a_3,a_4) = (a_1+a_3, -a_2, a_1, -a_1+2a_3, 0)$

T is a linear transformation

$$M = \begin{array}{c} \\ \end{array}\begin{array}{cccc} a_1 & a_2 & a_3 & a_4 \end{array}$$

$$\underset{5\times 4}{M} = \begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ -1 & 0 & 2 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

5 → # in co-domain; 4 → # in domain

→ Soln set for homo:

* $Z(T) = Z(M)$ → Span of ind. columns.

* Range(T) = Col(M)

$T(a_1,a_2,a_3,a_4) = M \begin{bmatrix} a_1 \\ a_2 \\ a_3 \\ a_4 \end{bmatrix}$

New: Find the image of

$T(1,0,0,0)$ go to formula, make $a_1 = 1$ and the rest zero. Or, 1st col. of M
$= (1,0,1,-1,0)$

$T(0,1,0,0) = $ 2nd column of M $= (0,-1,0,0,0)$

$T(0,0,1,0) = $ 3rd " " " $= (1,0,0,2,0)$

$T(0,0,0,1) = $ 4th " " " $= (0,0,0,0,0)$

$\{(1,0,0,0)\ (0,1,0,0)\ (0,0,1,0)\ (0,0,0,1)\}$ → They are independent → basis → can be written as span

↳ We call it : Ordered standard basis

Why ordered? I have many basis but the ones from $\mathbb{R}^4$ are called ordered standard basis $\{(P1)(P2)(P3)(P4)\}$
row1 row2 row3 row4

given any point in $\mathbb{R}$ say $(4,3,9,4)$ it is a linear combination
$4(1,0,0,0) + 3(0,1,0,0) + 9(0,0,1,0) + 4(0,0,0,1) = (4,3,9,4)$

$\{(\ ,\ )(\ ,\ )(\ ,\ )(\ ,\ )\}$ ⇝ If he gave M I can easily find the image of each
image 1st col. " 2nd " " 3rd " " 4th "

Q $T: \mathbb{R}^2 \to \mathbb{R}^4$ is a linear transformation → Range is space of specific points

$T(2,4) = (0,10,1,0)$

$T(0,1) = (1,0,2,4)$

Find $T(6,10)$ → Choose randomly but (①,②) we should have 2 things

Soln? next page

Know:

Assume T is linear transformation {if not linear comb, the result never true}

Image of linear combination of points ~~equal~~ in the domain = linear combination of image of the point.

$T(a_1Q_1 + a_2Q_2 + \cdots + a_kQ_k) = a_1T(Q_1) + \cdots + a_kT(Q_k)$

Linear combination of points — linear combination of image

Where $Q_1, \ldots, Q_k$ are points in the domain and $a_1, \ldots, a_k$ are some scalar

Q $T: \mathbb{R}^2 \to \mathbb{R}^4$ is linear combination

$T(2,4) = (0,10,1,0)$

$T(0,1) = (1,0,2,4)$

Find $T(6,10)$

$(6,10) = \alpha_1(2,4) + \alpha_2(0,1)$ → then find α.

$6 = 2\alpha_1 + 0\alpha_2 \to \alpha_1 = 3$

$10 = 2\alpha_1 + \alpha_2 \to \alpha_2 = -2$

$(6,10) = \overset{a_1}{3}(\overset{Q_1}{2,4}) - \overset{a_2}{2}(\overset{Q_2}{0,1})$

$T(6,10) = T(\overset{a_1}{3}(\overset{Q_1}{2,4}) + \overset{a_2}{2}(\overset{Q_2}{0,1}))$ by using prev. result.

$= 3T(2,4) - 2T(0,1)$ → linear comb. of (2,4) and (0,1) images

$= 3(0,10,1,0) - 2(1,0,2,4) = (-2,30,-1,-8)$ image of point in $\mathbb{R}^4$

In this question, I can calculate the image of any point in the domain.

↳ (2,4) (0,1) are independent in $\mathbb{R}^2$ ∴ they form a basis {(2,4)(0,1)} basis in $\mathbb{R}^2$

Span{(2,4)(0,1)} = $\mathbb{R}^2$ ∴ any point in $\mathbb{R}^2$ can be written as linear combination

* $T: \underset{x\text{-axis}}{\mathbb{R}} \longrightarrow \underset{y\text{-axis}}{\mathbb{R}}$

$T(x) = x^2$ linear trans.

$T(1+3) = T(4) = 16$ but $T(1) + T(3) \neq 16$ ⇒ Linear transformations have more properties than functions.

*Know:

Assume B is a basis for the domain of a linear transformation T.

If image of B is given then we can calculate/determine the image of all points in the domain.

Know:

Assume T is linear transformation {if not linear comb, the result never true}
Image of linear combination of points ~~equal~~ in the domain = linear combination of image of the point.

$T(a_1Q_1 + a_2Q_2 + \cdots + a_kQ_k) = a_1T(Q_1) + \cdots + a_kT(Q_k)$

Linear combination of points — linear combination of image

Where $Q_1,\dots,Q_k$ are points in the domain and $a_1,\dots,a_k$ are some scalar

Q $T: \mathbb{R}^2 \to \mathbb{R}^4$ is linear combination

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$T(0,1) = (1,0,2,4)$

Find $T(6,10)$

$(6,10) = \alpha_1(2,4) + \alpha_2(0,1)$ → then find α.

$6 = 2\alpha_1 + 0\alpha_2 \to \alpha_1 = 3$

$10 = 2\alpha_1 + \alpha_2 \to \alpha_2 = -2$

$(6,10) = \underset{a_1}{3}\underset{Q_1}{(2,4)} - \underset{a_2}{2}\underset{Q_2}{(0,1)}$

$T(6,10) = T(\underset{a_1}{3}\underset{Q_1}{(2,4)} + \underset{a_2}{2}\underset{Q_2}{(0,1)})$ by using prev. result.

$= 3T(2,4) - 2T(0,1)$ → linear comb. of (2,4) and (0,1) images

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In this question, I can calculate the image of any point in the domain.

→ (2,4) (0,1) are independent in $\mathbb{R}^2$ ∴ they form a basis {(2,4)(0,1)} basis in $\mathbb{R}^2$

Span {(2,4) (0,1)} = $\mathbb{R}^2$ ∴ any point in $\mathbb{R}^2$ can be written as linear combination

* $T: \underset{x\text{-axis}}{\mathbb{R}} \longrightarrow \underset{y\text{-axis}}{\mathbb{R}}$

$T(x) = x^2$ — linear trans.

$T(1+3) = T(4) = 16$ but $T(1) + T(3) \neq 16$ ⇒ Linear transformations have more properties than functions.

*Know:

Assume B is a basis for the domain of a linear transformation T.
If image of B is given then we can calculate/determine the image of all Points in the domain.

$T: \mathbb{R}^3 \longrightarrow \mathbb{R}^4$

$T(a_1, a_2, a_3) = (-a_2 + a_1, 0, a_3, a_1)$

Show T is linear combination

↳ We have to show that

$T(\alpha_1(x_1, x_2, x_3) + \alpha_2(y_1, y_2, y_3)) = \alpha_1 T(x_1, x_2, x_3) + \alpha_2 T(y_1, y_2, y_3)$

T is L.T iff each co-ordinate is a specific linear combination of the variables in the domain

Variables in the domain a_1, a_2, a_3

1st co-ordinate $-a_2 + a_1 \rightarrow -a_2 + a_1 + 0a_3$ ✓

2nd co-ordinate $0 \rightarrow 0a_3 + 0a_2 + 0a_1$ ✓

3rd co-ordinate $a_3 \rightarrow a_3 + 0a_2 + 0a_1$ ✓

4th co-ordinate $a_1 \rightarrow a_1 + 0a_3 + 0a_2$ ✓

Each co-ordinate is a specific linear combination of variables in the domain. ∴ It is a linear combination

What if it was $(-a_2 + a_1, 0, a_3, a_1^2)$

Last co-ordinate is not a linear transformation.

$a_1^2 = a_1 \times a_1 \rightarrow$ it is not a specific # ∴ NO.

What if it was $(-a_2 + a_1, 1, a_3, a_1)$

$1 \neq \alpha_1 a_1 + \alpha_2 a_2 + \alpha_3 a_3 \rightarrow$ We don't have a linear combination that gives 1 ∴ NO.

$T: \mathbb{R} \rightarrow \mathbb{R}$

$T(a) = a^3$

1 coordinate I can't have linear combination.

$T(x) = 5x$

it should be a linear comb. of this variable.

if $T(a) = 5a$ yes, it can be a linear combination

if $T(a) = 5a^2$ NO, it can't be a to a power.

if $T(a) = 5a + 2$ NO, it should pass across the origion

$T: R^{2\times2} \to P_3$

$T\left(\begin{bmatrix} a_1 & a_2 \\ a_3 & a_4 \end{bmatrix}\right) = (a_1+a_4)x + (a_1 - a_2)$

① Show T is linear transformation
② Find the corresponding fake linear trans. say F
③ Find Fake range
④ Find Fake zero's
⑤ Find range of T
⑥ Find zero's of T

* For Fake use M
* The range lives in P_3 ∴ when I solve I end with poly.
* $Z(T)$ lives in $R^{2\times2}$ (domain)
* $Fn = 0 \to$ zero's of fn.

$F: \mathbb{R}^4 \to \mathbb{R}^3$ change everything to language R^n to R^m then come back to the origional language

$F(a_1, a_2, a_3, a_4) = (0, a_1+a_4, a_1-a_2)$

Fake standard matrix representation

$$\text{Fake} = M' = \begin{array}{c} \begin{array}{cccc} a_1 & a_2 & a_3 & a_4 \end{array} \\ \begin{bmatrix} 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 1 \\ 1 & -1 & 0 & 0 \end{bmatrix} \end{array}$$ domain notation

$\text{Range}(F) = \text{Col}(M') = \text{Span}\{(0,1,1), (0,0,-1)\}$ → translate it

$\text{Range}(T) = \text{Span}\{(x+1, -1)\}$

ind # of fake = 2   ind # of range = 2

$Z(F) = Z(M')$ ⇝ Solve homogeneous system

$-R_2 + R_3 \to R_3$
$$\begin{array}{c} \begin{array}{cccc} a_1 & a_2 & a_3 & a_4 \end{array} \\ \begin{bmatrix} 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 1 \end{bmatrix} \end{array}$$

$a_1 = -a_4$ $a_3 \in \mathbb{R}$
$a_2 = -a_4$ $a_4 \in \mathbb{R}$

$Z(F) = \{(-a_4, -a_4, a_3, a_4) \mid a_3, a_4 \in \mathbb{R}\}$
$= \text{Span}\{(-1,-1,0,1)\ (0,0,1,0)\}$ → translate it

$Z(T) = \text{Span}\left\{\begin{bmatrix} -1 & -1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}\right\}$

ind # domain = ind # range + ind # zero's
2×2 = 2 + 2 ✓

$T: P_3 \longrightarrow \mathbb{R}^4$

$T(a_2x^2 + a_1x + a_0) = (a_0 - a_2, -a_0, 0, a_1 - a_0)$

Change to the Fake

$F: \mathbb{R}^3 \to \mathbb{R}^4$

$F(a_2, a_1, a_0) = (a_0 - a_2, -a_0, 0, a_1 - a_0)$ ↪ Solve then switch back

* live in $\mathbb{R}^4$ but never in $\mathbb{R}^3$

ind # domain = ind # R^4 (range) + ind # zero's

3 = 4 + ?

↳ ? = -1 NO!! we can't have -ve.

*Assume T is a linear transformation

$T: ? \longrightarrow 1$

① T is ~~onto~~ 1-1 iff ind # of zero's of T = # zero's

iff $Z(T) = \{0\}$

② T is onto iff Range(T) = co-domain

* Orthogonal

Q1 Q2

$(1,1,1,1) \cdot (-2,1,0,1) = (1)(-2) + (1)(1) + (1)(0) + (1)(1) = 0$

dot product

Q1, Q2 are called othogonal if $Q_1 \cdot Q_2 = 0$ and $Q1 \neq 0$ $Q2 \neq 0$.

Def:

Assume $Q_1, Q_2, .. Q_K$ are points in $\mathbb{R}^n$ st. non of them equal 0. We say $Q_1, .., Q_K$ are orthogonal if dot product of every two is zero.

5 points → dot of every 2 = 0 → then orthogonal

Big result:

If $Q_1, .., Q_K$ are orthogonal points, then they are independent

Converse need not be true. (if they are independent we can't conclude that they are orthogonal)

Ex. $Q_1 = (1,1)$

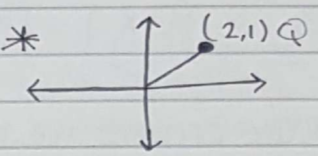
$Q_2 = (0,1)$

Q_1 and Q_2 are independent but $Q_1 \cdot Q_2 = 1$ not zero ∴ not orthogonal

$|Q|$ ⇝ Norm of Q or Length of Q

How Far Q from O ⇝ origion.

*



$|Q| = \sqrt{2^2+1^2} = \sqrt{5}$ ⇝ how for the point from the origion

* $Q = (2,2,1)$

$|Q| = \sqrt{2^2+2^2+1^2} = 3$

* $Q = (1,1,1,-2)$

$|Q| = \sqrt{1^2+1^2+1^2+(-2)^2} = \sqrt{7}$ $|Q|^2 = 7$ ⇝ No radical

Grand-Schmid algorithm

$D = \text{Span}\{\underset{Q_1}{(1,1,1,1)}\ \underset{Q_2}{(-1,-1,0,1)}\ \underset{Q_3}{(0,2,1,0)}\}$ ⇝ if they were orthogonal then the basis $\{Q_1, Q_2, Q_3\}$

Ind # of D = 3

* Will it be equal to $\mathbb{R}^4$? NO, it lives inside it.

* Find an orthogonal basis for D

Basis = $\{W_1, W_2, W_3\}$ where $W_1\ W_2\ W_3$ are orthogonal

$W_1 \cdot W_2 = 0$ $W_1 \cdot W_3 = 0$ $W_2 \cdot W_3 = 0$

Optional: Use row operations to get a reduced basis (easier calc.)

Always $W_1 = Q = (1,1,1,1)$ → dot

$W_2 = Q_2 - \frac{Q_2 \cdot W_1}{|W_1|^2} \times W_1$ → normal multiplication

$W_3 = Q_3 - \frac{Q_3 W_2}{|W_2|^2} \times W_2 - \frac{Q_3 W_1}{|W_1|^2} \times W_1$

$W_4 = Q_4 - \frac{?}{?} W_3 - \frac{?}{?} W_2 - \frac{?}{?} W_1$

$W_2 = \left[(-1,-1,0,1) - \frac{-1}{4}(1,1,1,1)\right] \times 4 = (-4,-4,0,4) + (1,1,1,1) = (-3,-3,1,5)$

Check $W_1 \cdot W_2 = 0$

dot p. dot p.

$W_3 = \left[(0,2,1,0) - \frac{-5}{44} \times (-3,-3,1,5) - \frac{3}{4} \times (1,1,1,1)\right] \times 44$

$= (0,88,44,0) + (-15,-15,5,25) - (33,33,33,33)$

$= (-48,40,16,-8)$

$W_3 \cdot W_2 = 144 - 120 + 16 - 40 = 0$ ✓

$W_1 \cdot W_2 = 0$ ✓

Basis = $\{(1,1,1,1)\ (-3,-3,1,5)\ (-48,40,16,-8)\}$

HW. $D = \text{Span}\{(1,4,1,1)(-1,2,3,1)(3,12,0,1)\}$ Find an orthogonal basis for D

Cramer

$3x+2y=10$
$-3x+y=11$ } 2x2 system of linear eqn

$$\begin{array}{ccc} A & X & B \\ \begin{bmatrix} 3 & 2 \\ -3 & 1 \end{bmatrix} & \begin{bmatrix} x \\ y \end{bmatrix} = & \begin{bmatrix} 10 \\ 11 \end{bmatrix} \end{array}$$

System has unique soln iff $|A| \neq 0$

To use cramer we must have nxn system of linear eqns where #eqn = # variables

Ex. 3x3 (3 eqn, 3 unknown)   3x10 (3 eqn, 10 unknowoun) X no cramer

① $A\begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix} = \begin{bmatrix} b_1 \\ \vdots \\ b_n \end{bmatrix}$ ② $|A| \neq 0$
the system must have a unique soln

co-eff. matrix

Use cramer to solve

$$\begin{bmatrix} 3 & 2 \\ -3 & 1 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 10 \\ 11 \end{bmatrix}$$

① $3\times1 - 2\times-3 = 9$ not zero   ② System of 2x2 ∴ I can use cramer

$|A| = 9 \neq 0$

replace it by const   Keep it the same

$$x = \frac{\begin{vmatrix} 10 & 2 \\ 11 & 1 \end{vmatrix}}{|A|} = \frac{10-22}{9} = \frac{-12}{9} = \frac{-4}{3}$$

Keep it   replace it by const.

$$y = \frac{\begin{vmatrix} 3 & 10 \\ -3 & 11 \end{vmatrix}}{|A|} = \frac{63}{9} = 7$$

$$\begin{array}{c} \\ \\ \\ \\ \end{array}\begin{bmatrix} x_1 & x_2 & x_3 \\ 1 & 2 & 0 \\ -1 & 4 & 3 \\ -1 & -2 & 10 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 4 \\ 2 \end{bmatrix}$$

→ if it was zero's, homogeneous
Since unique ∴ zero.

3x3

$|A| = 60$ Solve for x_3

No need to solve everything just choose one particular value.

$$x_3 = \frac{\begin{vmatrix} x_1 & x_2 & x_3 \\ 1 & 2 & 0 \\ -1 & 4 & 4 \\ -1 & -2 & 2 \end{vmatrix}}{|A|} = \frac{12}{60} = \frac{1}{5}$$

Change it (det)

Change it

$$x_2 = \frac{\begin{vmatrix} x_1 & x_2 & x_3 \\ 1 & 0 & 0 \\ -1 & 4 & 3 \\ -1 & 2 & 10 \end{vmatrix}}{|A|} = \frac{34}{60}$$

det

$$A^{-1} = \begin{bmatrix} \cdot\ \cdot\ \times \cdot \cdot \cdot \cdot \\ \vdots \quad xx \\ \end{bmatrix}$$

Remove from A the opposite (1,2) entery
→ (2,3) entery

Adjoint Method

Find A^{-1} using adjoint method

$A = \begin{vmatrix} 1 & 2 & 3 \\ -1 & 4 & 6 \\ -1 & -2 & 10 \end{vmatrix}$. Find A^{-1}

Find (1, 2) entery of A^{-1} ← No need to know all #s
↳ # in A^{-1} located in 1st row 2nd column

$$= \frac{(-1)^{1+2} \left| \begin{matrix}\text{Remove 2nd} \\ \text{row and 1 col} \\ \text{of A}\end{matrix} \right|}{|A|} = -\frac{\begin{vmatrix} 2 & 3 \\ -2 & 10 \end{vmatrix}}{|A|} = -\frac{26}{78}$$

opposite of (1,2)
(2,1)

4x4 → 16 det each 3x3
↳ everytime remove row and col

(2,3) entery of A^{-1}

$$A^{-1} = \frac{(-1)^5 \begin{vmatrix} 1 & 3 \\ -1 & 6 \end{vmatrix}}{|A|} = -\frac{-9}{78} =$$

LU Factorization

Q : $A = \begin{bmatrix} 1 & 3 & 4 \\ -1 & 2 & 0 \\ -1 & 4 & 10 \end{bmatrix}$ Find LU factorization of A

$A = L\,U$ → upper triangular

① lower triangular

② Invertible (non singular) $|A| \neq 0$.

We are not allowed to interchange rows, other operations are allowed.

↳ ∴ In this method we are not allowed to interchage rows. ⇝ interchange you get matri

[1] get U

* Change A to upper triangular → use row operations

$$\begin{bmatrix} 1 & 3 & 4 \\ -1 & 2 & 0 \\ -1 & 4 & 10 \end{bmatrix} \xrightarrow[R_1+R_3 \to R_3]{R_1+R_2 \to R_2} \begin{bmatrix} 1 & 3 & 4 \\ 0 & 4 & 4 \\ 0 & 7 & 14 \end{bmatrix} \xrightarrow{1/4 R_2} \begin{bmatrix} 1 & 3 & 4 \\ 0 & 1 & 1 \\ 0 & 7 & 14 \end{bmatrix}$$

$$\xrightarrow{-7R_2+R_3 \to R_3} \begin{bmatrix} 1 & 3 & 4 \\ 0 & 1 & 1 \\ 0 & 0 & 7 \end{bmatrix}$$ No need to cont. I got my upper triangular (U)

[2] get L

$I_3 \to \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ reverse all row operations $\xrightarrow{7R_2+R_3 \to R_3} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 7 & 1 \end{bmatrix} \xrightarrow{4R_2} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 7 & 1 \end{bmatrix}$

if A $4\times4 \to I_4$

if A $10\times10 \to I_{10}$

$$\xrightarrow[-R_1+R_3 \to R_3]{-R_1+R_2 \to R_2} \begin{bmatrix} 1 & 0 & 0 \\ -1 & 4 & 0 \\ -1 & 7 & 1 \end{bmatrix} \text{ (L)}$$

$$A \begin{bmatrix} & & \\ & & \\ & & \end{bmatrix} = \underbrace{\begin{bmatrix} 1 & 0 & 0 \\ -1 & 4 & 0 \\ -1 & 7 & 1 \end{bmatrix}}_{L} \underbrace{\begin{bmatrix} 1 & 3 & 4 \\ 0 & 1 & 1 \\ 0 & 0 & 7 \end{bmatrix}}_{U}$$

order is important.

writing a matrix as 2 multiplication lower and upper ~~tri~~ inv.

Solve the system $A\begin{bmatrix} X_1 \\ X_2 \\ X_3 \end{bmatrix} = \begin{bmatrix} 2 \\ 1 \\ 4 \end{bmatrix} \rightsquigarrow LU\begin{bmatrix} X_1 \\ X_2 \\ X_3 \end{bmatrix} = \begin{bmatrix} 2 \\ 1 \\ 4 \end{bmatrix} \rightsquigarrow L^{-1}LU\begin{bmatrix} X_1 \\ X_2 \\ X_3 \end{bmatrix} = L^{-1}\begin{bmatrix} 2 \\ 1 \\ 4 \end{bmatrix}$

$\rightsquigarrow U\begin{bmatrix} X_1 \\ X_2 \\ X_3 \end{bmatrix} = L^{-1}\begin{bmatrix} 2 \\ 1 \\ 4 \end{bmatrix} \rightsquigarrow \begin{bmatrix} 1 & 3 & 4 \\ 0 & 1 & 1 \\ 0 & 0 & 7 \end{bmatrix}\begin{bmatrix} X_1 \\ X_2 \\ X_3 \end{bmatrix} = L^{-1}\begin{bmatrix} 2 \\ 1 \\ 4 \end{bmatrix}$

↳ multiply and get 3x1 matrix

$\begin{bmatrix} 1 & 3 & 4 \\ 0 & 1 & 1 \\ 0 & 0 & 7 \end{bmatrix}\begin{bmatrix} X_1 \\ X_2 \\ X_3 \end{bmatrix} = \begin{bmatrix} n_1 \\ n_2 \\ n_3 \end{bmatrix}$ backward substitution

$7X_3 = n_3 \to X_3 =$

sub

$X_2 + X_3 = n_2 \to X_2 =$

$X + 3X_2 + 4X_3 = n_1 \to X_1 =$

Final Exam Review

$$* \underset{A}{\begin{bmatrix} 2 & a & b & 4 \\ 4 & c & d & 8 \\ 6 & 7 & g & h \end{bmatrix}} \sim \underset{B}{\begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}}$$

Coloumn space only from A

Row space either A or B.

$Row(A) = Span\{(1,1,1,1)(0,1,0,1)\}$

$(2,a,b,4) = \alpha_1(1,1,1,1) + \alpha_2(0,1,0,1)$

$\alpha_1 = 2$, $\alpha_2 = 4 - \alpha_1 \rightarrow \alpha_2 = 2$

then solve $(4,c,d,8) = 2(1,1,1,1) + 2(0,1,0,1)$

and for $(6,7,g,h) = 2(1,1,1,1) + 2(0,1,0,1)$

$* \; T: \mathbb{R}^3 \longrightarrow \mathbb{R}$ From here ind # $Z(T)$ + ind # range = ind # domain

2 + 1 = 3

$T(1,1,1) = 2$

$T(1,1,0) = 2$

$T(1,0,1) = 0$

$Range(T) = ?\ \mathbb{R}$ ↳ co-domain ↳ 1

Range ↳ 1 or 0

$* \; Z(T) \ni (1,0,1)$

$* \; T(1,1,1) - T(1,1,0) = 0$

$T((1,1,1)-(1,1,0)) = 0$

$T(0,0,1) = 0$

then check if $(0,0,1)$ and $(1,0,1)$ are independent.

$\therefore$ yes, $Z(T) = span\{(0,0,1)(1,0,1)\}$

OR

$T(x_1, x_2, x_3) = 2x_2$

$$M = \begin{array}{c} \begin{matrix} x_1 & x_2 & x_3 \end{matrix} \\ \begin{bmatrix} 0 & 2 & 0 \end{bmatrix} \end{array}$$

$Z(M) = \{(x_1, 0, x_3) \mid x_1, x_3 \in \mathbb{R}\}$

$= span\{(1,0,0)(0,0,1)\}$

* $D = \{A \in \mathbb{R}^{3\times4} \mid Rank(A) \leq 2\}$ Show it is not subspace

Test 3 axiums → One will fail

* $0 = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$ ✓

* Closure under addition → Choose 2 things that live in D

$V_1 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$ $V_2 = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$

$V_1, V_2 \in D$, add them $V_3 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$ ⇝ Rank (A) = 3

I'm out.

* $\{f(x) \in P_3 \mid f(-1) = 0 \text{ OR } f(2) = 0\} = D$

$0 \in D$ $f(-1) = 0$ ✓

Choose two things that live in D

$\left.\begin{array}{l} x+1 \in D ✓ \\ x-2 \in D ✓ \end{array}\right\}$ Add them $2x-1$ am I in D??

$\left.\begin{array}{l} f(-1) = -3 \neq 0 \\ f(2) = 3 \neq 0 \end{array}\right\} \therefore \notin D$

2nd axium fails

Least square method

① Find best fit of a plane of the form
$z = ax + by$ to the points $(1,1,1)$ $(-1,1,-1)$ $(0,2,6)$
② Explain the meaning of your answer

	Expected value of $ax+by$	given z-value
$(1,1)$	$a+b$	1
$(-1,1)$	$-a+b$	-1
$(0,2)$	$2b$	6

* Try solving the system → inconsistent $2b=6 \rightarrow b=3$. $a+b=1 \rightarrow a=-2$
$-a+b=-1 \rightarrow 2+3=5$ ✗

$$\underbrace{\begin{bmatrix} a & b \\ 1 & 1 \\ -1 & 1 \\ 0 & 2 \end{bmatrix}}_{A} \underbrace{\begin{bmatrix} a \\ b \end{bmatrix}}_{X} = \underbrace{\begin{bmatrix} 1 \\ -1 \\ 6 \end{bmatrix}}_{B}$$

inconsistent system → multiply by Transpose you get const. system

$$\underbrace{\begin{bmatrix} 1 & -1 & 0 \\ 1 & 1 & 2 \end{bmatrix}}_{A^T} \begin{bmatrix} 1 & 1 \\ -1 & 1 \\ 0 & 2 \end{bmatrix} \begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} 1 & -1 & 0 \\ 1 & 1 & 2 \end{bmatrix} \begin{bmatrix} 1 \\ -1 \\ 6 \end{bmatrix}$$

$a = 1$
$b = 2$

$\boxed{1}$ So, $z = x + 2y$

$\boxed{2}$

	Expected $x+2y=z$	given
$(1,1)$	3	1
$(-1,1)$	1	-1
$(0,2)$	4	6

$* (1-3)^2 + (-1-1)^2 + (6-4)^2$ = is minimum

given expected

→ Choose any other expected Ex. $4x+y=z$, $2x+2y=z$ ...
the difference is higher than $x+2y=z$ difference